Logic, Epistemology, and the Unity of Science 58

Eduardo Dorrego López
Elías Fuentes Guillén

Irrationality, Transcendence and the Circle-Squaring Problem

An Annotated Translation of J. H. Lambert's *Vorläufige Kenntnisse* and *Mémoire*

Foreword by José Ferreirós

Second Edition

Springer

Logic, Epistemology, and the Unity of Science

Volume 58

Logic, Epistemology, and the Unity of Science aims to reconsider the question of the unity of science in light of recent developments in logic. At present, no single logical, semantical or methodological framework dominates the philosophy of science. However, the editors of this series believe that formal frameworks, for example, constructive type theory, deontic logics, dialogical logics, epistemic logics, modal logics, and proof-theoretical semantics, have the potential to cast new light on basic issues in the discussion of the unity of science.

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The first edition contained numerous errors that originated during the production process of the book and for which the authors were in no way responsible. The second edition has allowed the publisher to correct these errors and the authors to make some improvements to the book.

To José Ferreirós,
for his teachings and friendship.

Foreword

The eighteenth century is a fascinating period, partly because of how different it is from the social and intellectual panorama that we are used to. The scientific disciplines as we know them did not exist, nor even a clear border between the "sciences" and the "letters" or humanities; there were no scientists, but "savants" or "erudites", philosophers who could devote themselves to many topics, including logic, history, languages, experiments, etc. Mathematics included questions concerning experimental science or even engineering (astronomy, mechanics, fortification, ballistics) and the place for science was not the university, but the Academies or even the salons of the aristocracy and haute *bourgeoisie*. Many of the "scientists" then had not enjoyed formal education, but were self-taught—such is the case of Johann Heinrich Lambert. These great differences with our specialized present are key to understanding the works included in this book, carefully prepared by Eduardo Dorrego López and Elías Fuentes Guillén.

Their author, who was born in Alsace[1] and became a member of the Academy in Berlin, offers us two entirely different writings—a carefully rigorous mathematical treatise in which the irrationality of π and e (and its powers) is proved (and where the conjecture that they are transcendental numbers appears for the first time in history); and a popular work meant to enlighten those who may feel the temptation to square the circle (by fractions). Both works were prepared in 1766 and 1767, although one was printed in 1768 in the *Mémoires* of the Royal Academy of Sciences of Berlin, while the other appeared in 1770, in volume II of the book *Contributions to the Employment of Mathematics and Its Applications*. The latter's popularizing goals are clear from the first page, and its ironical spirit is also evident; Lambert was happy to offer heuristic arguments and the broad outline of a possible proof of irrationality. We find, in these works, science as progress at the frontiers of knowledge, and as the enlightenment of the human mind: two fundamental aspects of the open yet deep conception that reigned in the eighteenth century.

[1] Mulhouse was a city associated with the Helvetic Confederation, a small Calvinist republic, no part of France.

Johann Heinrich Lambert is known among mathematicians for his work on number π, although he is usually regarded as a second-rate figure, little of his work being studied with care. In a paper published 40 years ago, Gray and Tilling discussed his figure, presenting him as a "little known" author who was "nevertheless interesting" and deserved a general overview.[2] But in his time he was seen as a first-rate author, who influenced people of the stature of Gauss and Kant, and whose work was admired for its profundity and breadth of knowledge. There were rumors that Euler himself might have abandoned the Berlin Academy because of Lambert (although in fact the reason seems to have been problems of management, not intellectual conflicts). Such facts alone should suffice to raise interest.

Let me offer details for two of them. Gauss's library included many of Lambert's works, which Gauss probably studied with care in his youth; the good knowledge he had of quantitative methods in mathematical physics, and of practical methods in the calculus (something that in his time was of great importance), are explained to a large extent by Lambert's influence.[3] Besides, it is likely that Lambert's ideas concerning non-Euclidean geometry guided the reflections of the young Gauss.[4] As for Kant, he had a correspondence with Lambert from 1765, when Kant regarded him as the "first genius of Germany", and his ideas were so relevant for the celebrated philosopher that he seriously considered dedicating the *Critique of Pure Reason* to him.[5]

The difficulty of evaluating Lambert, and the reason why he was somewhat forgotten in the nineteenth century, is precisely that he was so much a man of the Enlightenment. He was no specialist, but rather the opposite: a philosopher as much as a scientist, he contributed to all the sciences of his time; while active in the Academies of Munich and Berlin, he contributed to all the different «classes» or areas of work. It has been said that, for bad and good, Lambert was the perfect example of the eighteenth-century erudite, who wrote about God and the world, about all possible topics: mathematics, experimental science, philosophy, languages, history. An autodidact, independent, even stubborn in his way of thinking and his scientific choices, he was also a great promoter of German as a scientific and philosophical language; this had the side effect that some of his ambitious works were little known in other countries. We shall see that a fair evaluation of his best work could well require a vision so wide as his own.

Readers will find a lot of biographical material in Chap. 1 of his book, due to Eduardo Dorrego, so you can then admire and amuse yourself with the genial feats of this sage from Mulhouse. His first interaction with Frederick II of Prussia is

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[2] See (Gray et al. 1978).

[3] It is said that Lambert's tables of logarithms were carefully studied by him, and in fact they were related to his early interest in the distribution of prime numbers; these kind of tables were, at the time, an indispensable instrument for the working mathematician. See the paper "Logarithmentafeln—Gauss" "tägliches Arbeitsgeräth" by Karin Reich, p. 44 (in "Wie der Blitz einschlägt, hat sich das Räthsel gelöst". Carl Friedrich Gauss in Göttingen, ed. by Elmar Mittler, Göttingen, Bibliothek, 2005).

[4] (Abardia et al. 2012).

[5] He abandoned the idea due to Lambert's early death, in 1777, 4 years before the first edition.

astonishing, the impressions that great mathematicians like Lagrange had of him are of singular interest. Let us admit that Lambert was a strange character, one whom we would today rush to consider an Asperger, which might perhaps help understand some of his peculiarities. In his scientific trajectory he was an *Einzelgänger*, a lone wolf; often he preferred topics outside the mainstream, but even so he made important contributions. As a person, he was a man of Reason but also a devout protestant Christian (coming from a Huguenot family); an essentially independent writer, he was a free thinker in scientific and philosophical matters.

The best available description is perhaps that of John Heilbron, in phrases that Dorrego cites on p. 3:

> A self-taught polymath, he took as his main line the application of mathematics to physics and even to metaphysics. As a philosopher he worked out an epistemology similar to Kant's; as a physicist he sought effects linked by simple, general, and above all mathematical laws; as an experimentalist he advanced the quantitative study of photometry, pyrometry, hygrometry, and magnetism. He talked as an equal to Leonhard Euler and to Georg Brander, respectively the leading mathematician and the leading instrument maker in Germany. In a word, he was the perfect mathematical physicist: the mathematicians considered him an experimentalist with a "rare talent for applying calculation to experiments;" the experimentalists thought him a mathematician with an unusual understanding of the behavior of instruments. All of which (we are told) he accomplished by working from five in the morning to twelve at night, with a two-hour break at noon.[6]

The surprise of his contemporaries is easy to understand, because in the Enlightenment there were almost no mathematical physicists. It was unusual to encounter a very sound combination of experimental and mathematical abilities, in the style of a Galileo or a Newton—these were exceptions; for such a combination to become common, one had to wait until the nineteenth-century innovations in university teaching. Lambert, not content with such a prowess, added to all that the abilities of a real philosopher.

It will be worthwhile to mention some of his strictly scientific contributions, to show how they stood outside the main themes of his time. He developed photometric methods and introduced the notion of albedo, formulating also the Law of the Cosine in optics (instead of publishing about electricity, the topic of the day); in astronomy, he did important work on the comets (but nothing about rational mechanics). He designed a hygrometer, among other instruments, thanks to his great knowledge of instrument design; he contributed important works to cartography and the theory of maps, including the Lambert conformal projection. The reader probably knows of his anticipation of non-Euclidean geometry and his work on π—but he published no fundamental advance in the infinitesimal calculus.

It is relevant to speak a bit about Lambert's philosophy, which impressed Kant just when he was waking up from his dogmatic slumber, i.e., beginning his transition to criticism. The truth is that, so far as I can see, Lambert's philosophical ideas are poorly understood and still await a satisfactory interpretation. It is usually said that he was a follower of the Leibnizian-Wolffian school, but this kind of labeling helps very little with our understanding of even the rough traits of a way of thinking. Lambert

[6] (Heilbron 1982, pp. 66–67).

himself said that his philosophy joined elements of Wolff and Locke, but is it possible to create a coherent synthesis of empiricism and rationalism? Of course it is, in fact such a kind of combination seemed inevitable to those who worked in philosophy with their sight upon scientific methods. Here is an example of Lambert's approach: when in his *Neues Organon*, 1764, he studied the "simple concepts" at the basis of everything else, his analysis does not start with *a priori* ideas (as one expects of a rationalist), but rather has a phenomenological basis: experience gives the starting point, and by analyzing the contents of experience one finds the simple concepts. Lambert was convinced that such contents are not knowable apart from experience, but at the same time they have the form they have thanks to *a priori* elements imposed by the understanding. Little wonder that Kant felt great affinity with his ideas. If we are to simplify, considering the matter from a scientific perspective, one could say that Lambert's epistemology made room for a happy combination of Newtonianism and Leibnizianism.

As for mathematics, many were the relevant contributions due to him—infinite series, continuous fractions, work in geometry that prefigures Monge, hyperbolic trigonometric functions, conformal mappings, and so on. Consider, e.g., the contents of his *Contributions to the Employment of Mathematics and Its Applications* (Beyträge, 3 vols.): Vol. I (1765) deals with topics in practical geometry and trigonometry, as well as an interesting contribution to the theory of errors (systematic analysis of the reliability of observations and experimental results); Vol. II (1770) includes contributions to algebra and analysis, including the chapter on pi, but also studies of gnomonics and the Lunar tables of Mayer; and Vol. III (1772) discusses cartographic problems, the orbits of comets, architecture, and mortality rates. One may think, of course, that he was not a world-class mathematician like his colleagues Euler or Lagrange, that seems clear. As we saw already, it is interesting to think of Lambert as an applied mathematician and a mathematical physicist, more than a pure mathematician.

But it would be a mistake to conclude that he was not interested as well in "pure" issues—in fact, logical topics and foundational questions seem to be where his contributions are most original and forward-looking. Here we are interested above all in Lambert's conception of the foundations of arithmetic and of geometry, what might be called (in his time) the metaphysics of number and of space.

Lambert's colleague and great admirer Johann III Bernoulli—who edited his writings and was also Secretary of the Royal Academy at Berlin—believed that all of his works could be classified as belonging either to the physical-mathematical field, or to the area of logic.[7] He admired him as a philosopher with an extraordinary competence in logical thinking, «the greatest logician» of his century, and so he is considered even today.[8] How can we understand what Bernoulli says, knowing that

[7] Bernoulli's introduction to J. H. Lambert's *Logische und philosophische Abhandlungen*, vol. 1 (Berlin, 1782), p. X.

[8] In the *Britannica* online one can read: «The greatest 18th-century logician was undoubtedly Johann Heinrich Lambert.» See https://www.britannica.com/topic/history-of-logic/The-18th-and-19th-centuries (article by Hintikka and Spade, seen 18/08/2020).

his work includes a great amount of empirical work? It is compatible, I think, especially if we remember that the label "mathematics" was, at the time, wide enough to include all of the mathematical "sciences": photometry, cartography, architecture, and so on. Our puristic conception of mathematics is the daughter of the nineteenth century, once again, and is far from the vision of the Enlightenment.

Lambert's system of Logic was important at the time. Without knowing much of Leibniz's attempts, starting from the basic idea of an algebra of thought, Lambert developed a calculus that can be compared with Leibniz's, obtaining an elegant and efficient system of logic. The signs "=" y "+" were used in the manner of Leibniz and Boole, $+$ being a union of disjoint or exclusive concepts; yet this calculus was not extensional as in Boole, but intensional (terms denoted concepts, not things nor classes of individuals).[9] To express «All A is B», Lambert writes "$a = mb$", i.e., a known concept a is identical to the conjunction of b plus an indeterminate concept m (the idea is reminiscent of Boole). In this system, he carefully distinguished between known concepts, indeterminate concepts, and those that are strictly unknown. Even more interesting, he paid attention to relations (such as "the father of") and how their introduction would affect logic; he introduced a way of expressing relational notions by means of functions ("$i = \alpha :: c$" indicates that i is the result of applying the unary function α to concept c). It would be interesting to know whether authors like Frege or Dedekind had read Lambert, given that the introduction of relations and the incorporation of the mathematical idea of function to logical theory were key innovations towards the end of the nineteenth century.[10]

Be that as it may, Lambert's logical calculus was not decisive for the next advances of mathematical logic, despite his achievements and even though he influenced Moritz W. Drobisch (and through him, perhaps also Boole). Let me suggest that the best results of Lambert's extensive dealing with Logic may have been the impact it had upon some of his contributions to the foundations of mathematics: his conception of geometry, axiomatic and innovative, and the unusually rigorous approach of his work on π.

As regards the number π, the reader will find all the details in the study and translations featured in this book. Suffice it here to call attention to a few points: Lambert conceived a logically rigorous proof of irrationality, with almost no gaps, at a time that was not precisely characterized by attention to rigor—this was 60 years before Cauchy. And he took the step of distinguishing between algebraic and transcendental irrationals, a distinction that would mark the future of the topic, but which was not adopted by other mathematicians (except for Legendre) until long after; starting around 1840, it was taken up by figures like Liouville, Dirichlet, and a few others. Today we realize how incomplete was the conception of real numbers that was usual around 1800: they were only thinking of quadratic irrationals, numbers

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[9] *Sechs Versuche einer Zeichenkunst in der Vernunftlehre* (1777) "Six attempts of a symbolic art in logic". Included in *Logische und philosophische Abhandlungen*, vol. 1 (Berlin, 1782).

[10] The syllogistics of Aristotle is severely limited because it works only with monadic predicates (like "is mortal" or "is a mammal") and cannot deal with the logic of relations. The notions of relation and function are crucial innovations of the eighteenth and nineteenth centuries, see, e.g., Ernst Cassirer's book, *Substance and Function* (1910).

such as $\sqrt{2}$, or $\sqrt{2+\sqrt{3}}$, or cubic roots $\sqrt[3]{5}$, but Lambert took a giant step towards a correct conception of the system of real numbers. It was substantial progress to realize that there might be a whole class of irrational numbers beyond the algebraic ones, Lambert was a pioneer here. As regards the logical rigor of his proof, there are indications that it might have influenced other key authors, as might be the case of Gauss (think of his rigorous proofs of the fundamental theorem of algebra in 1799, and in 1816) or Bolzano (think of his papers of 1816 and 1817, including the famous work on the intermediate value theorem).

Considering geometry, an expert like V. de Risi underscores how Lambert pioneered the reintroduction of a strictly axiomatic conception of the subject, after two centuries (seventeenth and eighteenth) when it had become standard to think of the deductive edifice of geometry as based upon *definitions*. (The first truths follow from definitions, e.g., the circle was defined genetically, as the result of moving a segment around one of its extremities, fixed, back to its initial position; from which it was "immediately" derived that all of the radii of a circle are equal, and that one can describe a circle around any point as center, with radius equal to any given segment (Euclid's postulate).) On the contrary, Lambert insisted in his "Theory of Parallels" (written 1766, published by Bernoulli in 1786) upon the idea that the key assumptions that determine the content of the discipline are contained in the postulates or axioms of geometry, and he went so far as to suggest the idea of a strictly formal derivation of the theorems of geometry. This was an anticipation of the famous idea of Pasch and Hilbert, by more than a century! Here is the key passage from §. 11 of the "Theory of Parallels":

> In the first part of this question [*whether it* (the parallel postulate) *can be derived in proper order from the* (other) *Euclidean postulates together with his other axioms*], one can abstract from everything that I earlier called *representation of the thing*. And since Euclid's *postulata* and other axioms have been expressed in words, it can and should be demanded that the proof never appeal to the thing itself, but that the proof should be carried out purely symbolically—when this is possible. In this respect, Euclid's *postulata* are as it were like so many algebraic equations which one already has in front of oneself and from which one is to compute x, y, z, etc. without looking back to the thing itself.[11]

Let me end here, for I think we have insisted enough on the stature of the author, a scientist and thinker of first rank, and the interest of his works published in this book. The reader will find an excellent translation of both of them, with detailed and enlightening introductory studies. It is a novelty to find published together the popular essay from *Beyträge* and the erudite one from *Mémoires*, something that was only available in Spanish before. I believe this is a great idea, making possible a dual approximation to a fundamental issue in mathematics.

Incidentally, I would like to add that both works have been confused in the past, which caused some incorrect (negative) opinions about the nature of Lambert's

[11] Lambert translated in (Ewald 1996, p. 166).

mathematical work on π. The reader of this work will no longer fall into such confusions.

Seville, Spain José Ferreirós

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Eduardo Dorrego López studied Mathematics (minor in Pure Mathematics) at the University of Santiago de Compostela (USC), where he also obtained his Master's degree in the Department of Algebra. He got enrolled into the "Doctorate in Mathematics" program at IMUS (Seville) obtaining his Ph.D. in History of Mathematics under the supervision of Prof. José Ferreirós (2021). His research focuses on the 18th- and 19th-Century mathematics and, more specifically, on the development of irrational quantities and the circle-squaring problem. He has carried out research stays in Oxford and Seville working primarily on Lambert and Lagrange. He has published a book (in Spanish) on Lambert and his contribution to the irrationality of pi and the circle-squaring problem (with Elías Fuentes; College Publications), as well as a book chapter (in Spanish) on Lambert's work on non-euclidean geometries (with José Ferreirós; forthcoming). He is also a high school mathematics teacher.

Elías Fuentes Guillén is a researcher at the Institute of Philosophy of the Czech Academy of Sciences and head of the GA ČR Junior Star project "Normalisation and Emergence: Rethinking the Dynamics of Mathematics". Previously he held post-doctoral positions at the Department of Mathematics of the Faculty of Sciences at UNAM and the Institute of Philosophy of the Czech Academy of Sciences. His research focuses on the transition from mathematical practices that were common in the late 18th century to practices that emerged in the second half of the 19th century, as well as on the work of Bernard Bolzano. His recent publications include the book *Matematické dílo Bernarda Bolzana ve světle jeho rukopisů* (Nakladatelství Filosofia, 2023), a chapter for Springer's *Handbook of the History and Philosophy of Mathematical Practice* (2022) and "The 1804 examination for the chair of Elementary Mathematics at the University of Prague" (with Davide Crippa; *Historia Mathematica*, 2021).

Part I
Eduardo Dorrego López

Chapter 1
Johann Heinrich Lambert: A Biography in Context

Lambert is an interesting case. A self-taught polymath, he took as his main line the application of mathematics to physics and even to metaphysics. As a philosopher he worked out an epistemology similar to Kant's; as a physicist he sought effects linked by simple, general, and above all mathematical laws; as an experimentalist he advanced the quantitative study of photometry, pyrometry, hygrometry, and magnetism. He talked as an equal to Leonhard Euler and to Georg Brander, respectively the leading mathematician and the leading instrument maker in Germany. In a word, he was the perfect mathematical physicist: the mathematicians considered him an experimentalist with a 'rare talent for applying calculation to experiments;' the experimentalists thought him a mathematician with an unusual understanding of the behavior of instruments. All of which (we are told) he accomplished by working from five in the morning to twelve at night, with a two-hour break at noon.

—J. L. Heilbron, *Elements of Early Modern Physics.*

1.1 Introduction

The sixteenth century in Europe began with an event that after two centuries would lead to a change in the way of conceiving things, and a desire on the part of the people to break the shackles of oppressive and impoverishing intolerances. On October 31, 1517, on the door of the castle church of Wittenberg, Martin Luther nailed, according

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to tradition,[1] his famous Ninety-five Theses in which he highlighted the need to end the corruption of a church entity that used indulgences as a bargaining chip. These practices, already denounced before, became notorious at this time in which, for example, one could free oneself from purgatory in exchange for participating in the construction of St. Peter's Basilica.

With the support of the printing press —something that at first did not worry Rome— word quickly spread, becoming a problem even for Luther himself, since it led to violent revolts that he himself condemned.[2] There was now a split in the group of believers who considers a reform to be necessary, thus bringing forth Protestantism. As a reaction to this situation, there was convened a council in Trent that ran interruptedly between 1545 and 1562, where possible counter-reforms of the Church in the face of this new movement were discussed. The tensions that were more than evident led to various conflicts, provoking in this latter year the begining of the Religious Wars between Catholics and Calvinist Protestants in France. The Protestant branch in France had its origin in the ideas of John Calvin, a theologian who followed Luther, and were called the Huguenots. The Massacre of St. Bartholomew in 1572 where thousands of Huguenots were killed marked the height of barbarism in France.

With the turn of the century, the enormous fragmentation caused by the events of the previous century was leading to a resurgence of conflicts throughout the continent, which in practice led to a harsh persecution of Protestantism. For example, the Netherlands, which had been under Spanish rule since the 16th century, largely adopted the Calvinist religion after the Reformation. The religious tug-of-war stemming from this difference with their Catholic rulers led to wars and peace treaties that brought to light the clear difference between the north and the south of the region. While the north, led by William of Orange, continued to claim some independence, several southern provinces in present-day Belgium eventually ended up being annexed to the Spanish state. This breaking point is known as the Union of Arras of 1579 in which, among other things, the Catholic religion was established as the one and only religion, and the persecution of Calvinism was put into full effect.

From this area of the southern Netherlands came the Lamberts,[3] specifically from Wallonie (Wallonia), one of the three parts into which present-day Belgium is divided and which at that time was part of the Holy Roman Empire.[4] Like many others, they ended up escaping Catholic persecution in Lambrecht, a town located about

[1] Although it is very likely that this was the case due to the testimonies we have from his collaborator Melanchthon and his secretary Georg Rörer, it is something that specialists still do not take for granted (see Roper (2017, pp. 11, 451 notes 2 and 3)).

[2] It should be noted that along with the revolt against the church there was also a peasants' war, whom Luther initially supported, although he later retracted from this position cf. Roper (2017, Chap. 12). I am grateful to one of the anonymous reviewers for bringing this point to my attention.

[3] The origins of the Lambert family of Mulhouse are studied in Mieg (1939, pp. 27–30) by the historian and genealogist Philippe Mieg (if I have been able to consult this article it has been thanks to the kindness of Eliane Michelon from the *Archives de Mulhouse* who disinterestedly allowed me access to it); a summary can be found in Jaquel (1977, pp. 133–135). On the other hand, with regard to the historical context, I depended entirely on Parker (1997), Bergin (2001) and Oberle (1985).

[4] On the Holy Roman Empire see Stollberg-Rilinger (2018).

70 km from Heidelberg, the capital of the Calvinist Lower Palatinate and one of the main centers of reformed religion in Europe of that time. Apparently in this locality there was a colony of French-speaking Calvinists —refugees themselves from the Netherlands and Belgium— who had settled in 1568 in the disused buildings of the old convent, and it is possible that the Lamberts, who had also arrived in the second half of the century, found comfort and support among them. In fact, both Lambert's great-great-grandfather, Jean Colin Lambert, and his great-grandfather Jean Nicolas Lambert (called Colin) were born there, although it did not take long for the complicated situation in the north to make their time there dangerous.[5]

Years later in Bohemia (present-day Czech Republic) —a predominantly Protestant region— the conflict with the ruling Habsburg Catholics culminated on May 23, 1618 when two ministers of the Emperor Matthias and one of their secretaries were thrown out of a window.[6] Anticipating what was coming their way, they asked for help from the leader of the Protestant Union, Frederick of the Palatinate, and offered him the crown of Bohemia, deposing Ferdinand, who a year later was elected Holy Roman Emperor. From the Protestant point of view, control of Bohemia was no mere bagatelle; it was in fact vital in order to prevent the end of religious freedom in the Empire. However, this vision was also shared by the Catholics, who sent a retinue of Spanish troops to the Palatinate in 1620 to thwart an attack from the rear and secure their position in the process.[7] The proof of the realness of these visions is that afterwards, after the Catholics ended the Bohemian revolt at the Battle of the White Mountain in Prague on November 8, 1620, the re-catholicization swept the Palatinate, spreading throughout the Empire. Perhaps the capture of Heidelberg, once the capital of European Protestantism, by the Catholics in 1622, was the most representative symbol of the clear defeat of the Protestants.

The situation was therefore no longer safe, so the Lamberts packed up their belongings and headed to Mülhausen. It is more than plausible that they had not chosen the new destination by chance, since it was a small Calvinist Republic of northern Alsace, allied since 1515 with the Helvetic Confederation (Switzerland; in fact, it was one of the few parts of Alsace that would not be annexed to France after the Thirty Years' War). Although the Edict of Nantes of 1598, which ended the French Religious Wars, had established a certain tolerance towards Protestants, «it had never been applied in Alsace»,[8] so this hostile geographical environment made Mülhausen a small, Calvinist paradise. In fact, throughout the seventeenth century it had served as a refuge for many Protestants who came from various parts of France, such as Lorraine,[9] although the poor economic situation made it difficult to grant the right of the bourgeoisie, which allowed the right to exercise trade.

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[5] The colony itself would be dissolved in 1623 with the arrival of Spanish troops.

[6] The so-called «Defenestration of Prague».

[7] Key step towards the Thirty Years' War Parker (1997, p. 76).

[8] Oberle (1985, p. 12) who quotes Pfister, *L'Alsace et l'Edit de Nantes* in Revue historique, 1929 (pp. 217–240).

[9] It is possible that the tradition of placing the Lamberts as refugees from Lorraine (or more generally from France) comes from here, a tradition that Matthias Graf, pastor of Mülhausen and author of a reference biography on the Swiss published in 1829 on the occasion of the centenary of his birth

Therefore, it could not have been easy for Jean Nicolas Lambert, the great-grandfather of our savant, to obtain this right. He had come to Mülhausen in 1624 with his widowed mother Marie Marx and his uncle (and tutor) Jean Nicolas de Cornesse —a native of Cornesse in Wallonie who had been burgomaster of Lambrecht in the Palatinate— and he did not obtain this right until eleven years later in 1635. And thus Jean Nicolas, who was a master baker, became an «échevin»[10] in 1655, a role that he passed on to his son Jérémie (1660–1733) who would eventually become a tailor, a profession that would be inherited by his son and Lambert's father, Lucas Lambert (1699–1747).[11] Poor business decisions made after 1660 forced Jean Nicolas to gradually sell most of his assets, which is most likely what motivated the Lamberts to leave Mülhausen for the Palatinate in 1671; however, after the death of the head of the family and the destruction of Lambrecht by French troops, his widow and his son, Jérémie, returned around 1689 to stay permanently. Mülhausen remained a good option in a Europe that was still dangerous, despite it having left the Thirty Years' War behind with the signing of the Peace of Westphalia (1648). Eventually, in 1685 Louis XIV revoked the Edict of Nantes, giving a renewed vigor to the persecution of the Calvinists;[12] a persecution that in reality had already been internationalized —and increasingly so— with the Edict of Restitution of 1629, which basically prohibited all Protestant sects except Lutheranism.[13]

All these conflicts together had drawn a landscape in Europe that was, especially in Germany, quite bleak. But in the midst of this panorama, there occurred a change in the mentality of certain social groups —thanks to discoveries by such figures as Galileo, Kepler, Descartes or Newton, and to the impact of works by Locke, Bayle or Leibniz— a change whose objectives would be to break from dogmas, use reason as a guide, «the enlightenment of all human beings as a fight against superstition and their education to public application and utility»:[14] the Age of Enlightenment was born.[15] It was in this context that Lambert's life took place.

(in Huber et al. (1829)), even situated within Lambert's own family. An example is found in the biography of Formey on the occasion of his *Eulogy on Lambert* Sheynin (2010, p. 137), or in the article by Scriba (1973, p. 595) (to whom Sheynin (2010, p. 5) refers as Lambert's modern biographer). Furthermore, the reader studying Lambert's biography will notice how his family origins are usually placed among refugee Huguenots, although they were not really French. Philippe Mieg values this bibliographic tradition, finding in it some confirmation of what was already stated above to the effect that Lambert's family had had contact with a Huguenot colony at Lambrecht (see Mieg (1939, pp. 26, 29)).

[10] In the Larousse dictionary: «In the Middle Ages and under the old regime, municipal magistrate [«magistrat», defined as «a character invested with important public functions»] in the cities of northern France, who assists the mayor [«maire», defined as «the first of those magistrates»]».

[11] See Sitzmann (1909, p. 92) (based on Mieg (1939) and Jaquel (1973), the author must have been wrong in saying that he obtained the bourgeois right in 1645).

[12] It will not be until 1787 that a measure of tolerance for them is restored in France (see Blanning (2000, pp. 135, 151)).

[13] Parker (1997, pp. 127, 128).

[14] Hermann (1988, p. 123).

[15] I use this term in a broad sense and without entering into terminological distinctions based on what place —France, England, Germany, etc.— one wants to focus on, as well as knowing that

1.2 Early Years (1728–1746)

Johann Heinrich Lambert was born in Mülhausen on August 26, 1728, four years after his parents, Lucas Lambert and Elizabeth Schmerber, married.[16] The family's financial situation was difficult; the modest salary of his father, who continued the tailoring profession, together with the need to support a large family of ten children, three of whom died at a young age,[17] forced a standard of living far from comfort. However, his parents did not neglect their children's need for basic education:

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normally and without a clear consensus, it is framed in three different periods: «the long» (1688–1815), «the strict» (1700–1800) and «the short» (1715–1789) «Enlightenment» (the main source used for the historical period referred to has been Blanning (2000)).

[16] It must be said that the date of birth is not known for sure because at that time in Mülhausen there was no registration of birth, only of baptism (Lambert is baptized on August 29) (Jaquel 1973, p. 102). Although Jaquel says that the 26th is usually adopted as his date of birth, one finds in some biographies of the XVIII other accounts, such as the 29th in Barlow (1814), or the 28th (of April!) in Hutton (1815, p. 710). As for Lambert's origins, in Lambertian historiography there is confusion about the nationality of the savant since on many occasions he is presented as French, German or Swiss. The Lamberts' regions of origin were predominantly French-speaking, and they seem to have had contact with French communities in Lambrecht; furthermore, much of Alsace was dominated by the French (but not Mülhausen), it was completely French between 1798 and 1871 when it passed into German hands, and it was again from 1918, something frequently used to classify Lambert as French. On the other hand, on his father's side he is of German origin, or at least until his great-grandfather arrived in Mülhausen from regions dominated by the Holy Roman Empire of the German Nation, and spent the last part of his life (12 years) in Germany, where he found his place (also between 1871 and 1918 Mülhausen came under German dominance, which also caused some to view Lambert as a German, in addition to the fact that his maternal language was Alsatian, a German dialect). The same would apply to the origin of his mother, her great-grandparents having been mainly Mulhousians and also Germans. Lastly, approximately the first 15 years of his life were spent in his hometown, at that time part of the Confederacy Helvetica (Switzerland). Jaquel (1973) analyzes the case in detail and comes to the conclusion that, although Mülhausen's relationship with the Confederation was variable and the simplest and most rigorous at the same time would be to consider Lambert as Mulhousian, the most common and natural practice is to consider Mülhausen as a Swiss city. In this way, the most natural thing would also be to classify Lambert as Swiss. In this same line: Knobloch in Begehr et al. (1998, p. 5) considers Lambert Swiss, since Mülhausen in those days belonged to Switzerland until it was annexed to France; Rudolf Wolf (1816–1893) in his biography on Lambert translated from German in Sheynin (2010) adds (p. 150) that Lambert:

> invariably considered himself a Swiss and until he earned any scientific title his contemporaries called him *Mülhusino-Helvetus*. I cannot therefore hesitate to describe that great thinker as a Swiss scientist.

Cajori (1927, p. 129 note 5) and Gray (2007, p. 84) present him as Swiss without further details; Calinger (2016) refers to him as Swiss-German (p. 643), although he clarifies that his hometown was in Switzerland (p. 427). In p. 558 note 22 he is clearer and speaks of him as Swiss.

[17] The data in Jaquel (1973, p. 102). In Klemme et al. (2016, p. 451) it is said that Lambert had four siblings, but it is also said (as elsewhere) that his family arrived at Mulhouse in 1635 as escaped refugees from Lorraine. By the way, and in line with what has just been said in the previous note, the title of this work shows the preference of the authors towards the German nationality of Lambert, although it would have to be said that considering him a German philosopher is a natural historiographic tendency since «he endeavored to develop a German philosophical language, and used only German in his impressive philosophical works» Jaquel (1973, p. 104).

Lambert attended the school in his city —revealing himself a diligent and gifted student— where he received elementary training in French, Latin and other subjects. But, at the early age of 12 (Fig. 1.1), he had to leave his studies to help his father with tailoring. Even in that very short period of time though, he showed signs of a strong inclination towards study, something unusual for his age, even more so if one takes into account that he did not come from a family of intellectuals.

Lambert devoted the little free time he had left after helping his parents to reading. Even at night while the rest of his family was asleep, he studied by candlelight. His mother, possibly worried about his lack of rest, sometimes took them away, but he got more by selling small drawings that he made. His strong and surprising dedication, together with the references that the teachers gave of him, made his father considers the idea of letting his son leave the tailoring business and dedicate himself to what clearly appeared to be his passion: study.

At the time, those who had the means to attend university were first required to go through some philosophical studies (i.e. to attend the so-called Faculty of Philosophy), which had a foundational role and granted access to the Faculties of Theology, Medicine and Law, which were the only studies offered.[18] Formey in his *Eulogy on Lambert* (1780) summarized the intellectual environment in our savant's hometown by noting that:

> It is appropriate to mention that in those times the number of men of letters in Mulhouse was restricted to half a dozen theologians since it was thought that there did not at all exist any other science except theology or otherwise that only theologians were able to develop sciences.[19]

Given that Lambert's family was very religious, it is understandable that they made the decision, following the advice of their son's professors, that he should study theology. His father tried to get financial help for his son, but to his despair, he was not able to find anything. Finding that their circumstances were not changing, Lambert had to continue tiptoeing between tailoring and studying.

Despite his disappointment, he did not give up, and continued to use the little free time he had to study whatever fell into his hands: two books on arithmetic and geometry which helped to consolidate his basic mathematical culture (one loaned

[18] See Ferreirós (1995). These philosophical studies included history, mathematics, philosophy in the strict sense, physics, philology etc., of course, as they were understood at that time. To give just one example, in the eighteenth century physics was understood as «the science that teaches us the reasons and causes of all the effects that Nature produces» (Rohalt cited in Hankins (1985, pp. 10–11)), and so medicine, among others, was understood as part of physics. In fact, in the seventeenth century the physicist and the doctor were the same thing, and even today in certain languages such a connection can be traced in the designated words for a doctor (in English «physician» is defined in the Cambridge Dictionary as «a medical doctor, especially one who has general skill and is not a surgeon»).

[19] Sheynin (2010, p. 138). Texts referenced in Sheynin (2010) are: the *Eulogy on Lambert* (1780) by Johann Heinrich Samuel Formey (1711–1797), perpetual secretary of the Berlin Academy of Sciences among whose duties was to make the obituaries of the deceased members; and an 1860 biography of Johann Rudolf Wolf (1816–1893), professor of astronomy in Zurich. In what follows, and whenever appropriate, it will be made explicit which of the two is being referred to.

Fig. 1.1 Lambert in his youth (from the web: *Johann Heinrich Lambert (1728–1777) Collected Works-Sämtliche Werke Online*). As for Lambert's portrait, see Appendix A

by one of his colleagues and the other borrowed from a worker hired by his father who was surprised by the dedication to reading shown by the young Lambert). In conjunction with his readings, he had a local teacher helping him with French and Latin for free, and Heinrich Reber, the city's scribe, after seeing the quality of his calligraphy, hired him in his office as a copyist. Reber was a key figure for Lambert in these early years for the support he gave him and for his recommendations.

After a stint as a copyist, at the age of 15 and on Reber's recommendation, he went to work as an accountant in the steel industry in Seppois in northern Alsace. There he perfected his French and, among other things, followed with great attention the course of the comet of 1744, which would later motivate his work on these subjects.[20] From now on Lambert would not look back. He was to put aside the family tailoring once and for all, and was to dedicate the next decade to training from all angles without neglecting any branch of knowledge, which was to make him the polymath scientist that he was.

1.3 Epoch of Learning (1746–1756)

After returning from Seppois in 1746, the 18-year-old Lambert moved to Basel to become, on Reber's recommendation, secretary to the Swiss philosopher Isaac

[20] Listed as a Great Comet, it was especially bright and spectacular, eventually developing a 6-tailed fan after reaching perihelion.

Iselin,[21] who at that time was editor of a political newspaper. In a letter dated December 6, 1750, Lambert talked about his work in Basel:

> About four years ago I had basically learned Latin and French and then the late city scribe Reber recommended me to Dr Iselin in Basel to be helpful to him with his correspondence and newspaper articles.[22]

Lambert's personality was far from charming and he had a rather strange character, but:

> on the one hand, his extraordinary ability and, on the other, the incorruptible rectitude of his spirit, won him reliable allies capable of appreciating his virtues and forgiving his temperament.[23]

Iselin was one of those who, along with Reber and his former teachers, saw in Lambert a burning desire to learn. He came to develop great esteem for him. He educated him during the day and allowed him to attend his lessons, but his deeply ingrained autodidactic propensity made him prefer to take refuge in his own self-acquired books before attending his classes. This same letter shed light on his early influences:

> In that capacity hardly half a day am I occupied so that I have got myself some books for learning the elements of wisdom. I have understood at once that my first efforts should be directed at perfecting my knowledge and making myself happy. However, I also understood at once that naturally depraved intentions cannot be improved without freeing the mind from prejudices and properly enlightening it. That was therefore my first reference point, and I find those rules, which are very useful for cognizing the mind itself and its faults and for investigating the truth, in the writings of Wolff on the power of the human mind, of Mallebranche on the investigation of truth, and of Locke's thoughts on the human mind. All this is above all revealed in the mathematical sciences and especially in algebra and mechanics which provided me with clear and thorough examples enabling me to confirm the previously learned rules and to transform them, so to say, into my own flesh and blood. Until now, I have found no reason to regret my efforts since now I am able all the better to learn other sciences easier and more thoroughly and since I ought to teach others, to explain everything much better and more skilfully.

In addition to the importance that he gave to mathematics as a paradigm of the philosophical ideas that these books taught him and as a fundamental tool for the other sciences,[24] this small extract from the letter verifies that his mind was indeed free of prejudices. Although for a person of that time with intellectual leanings, these three works, among others, were almost required readings, it is also true that Wolff

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[21] Born in 1728, he studied law and philosophy at the Universities of Göttingen and Basel (he became professor of law at this latter university). A respected man, he was one of the founders of the Helvetic Society. He died in 1782 as a permanent member of the Berlin Academy.

[22] Wolf in Sheynin (2010, p. 151).

[23] Juan Arana in Lambert (1765/1767, p. 200).

[24] Here one can already see, as will be seen later, his intertwined vision of the different parts of science.

Fig. 1.2 Lambert in his maturity stage. Portrait made by illustrator G. Dantzer around 1850 (availabe at http://ark.bnf.fr/ark:/12148/cb41920778f)

(together with Malebranche) and Locke represented the two opposite currents of the theory of knowledge: rationalism vs empiricism.[25]

Despite the affection he had for Lambert, Iselin made his desire to retain him dependent on the search for a possibility that would provide a good option for his development as a scientist. This is how our Mulhoussian came to travel in 1748 at the age of 20 to Chur, the capital of the Canton of Grisons in Switzerland, to become the private tutor of three young relatives of Count Peter von Salis (Fig. 1.2). Von Salis, who at that time was 80 years old, in addition to being Count of the Holy Roman Empire of the German Nation, had been an ambassador in London and one of the negotiators of the Peace of Utrecht. He was therefore an important, influential, and cultured man —a possessor of a great library that would open the doors for Lambert to a broader and deeper study with which he would define his scientific and philosophical thought.

His work in Chur consisted of personally educating the 11-year-old count's grandson Antoine de Salis, his 11-year-old cousin Baptista, and another 7-year-old relative named Johann Ulrich von Salis. Over the next decade he would instruct them in languages, mathematics, geography, history, and catechism (the von Salis family were very devout, a quality that Lambert shared and would uphold). The free time available to him was devoted to studying in the library of his host, and like one who tries

[25] He won't marry either of them, so to speak, but in fact he will marry the two of them together (see Gray et al. (1978)).

to quench his thirst after a long journey without water, he indiscriminately embraced physics, astronomy, mathematics and mechanics, as well as theology, metaphysics, and even poetry, making regular astronomical observations and building his own instruments for experiments.

With this background, he began to develop his own reflections, which he wrote down every month, starting in 1752, in his *Monatsbuch*, a scientific diary which he would continue writing up until his death.[26] It was also at this time that he came into contact with the academic world. Together with the support of his sponsor, his rapid progress and the knowledge he had acquired led to him being appointed in 1752 to the Chur Literary Society and later to the Swiss Scientific Society based in Basel. At the request of this institution, he would make several meteorological observations that would be materialized in various publications in *Acta Helvetica*, the Society's journal. It was in this journal, specifically in volume 2, that he was to publish in 1755 his first article *Tentamen de vi caloris, qua corpora dilatat ejusque dimensione*[27] which dealt with heat, one of the main topics of study of physics of the time, along with that of light, electricity and magnetism.

On September 1, 1756, after eight years at the von Salis' home —a key period for Lambert's training that allowed him to more than make up for the educational deficiencies that he had suffered due to lack of resources when he was younger— he embarked with Antoine and Baptista, who were already 19 years old, on an academic trip through Europe in which he would visit the main intellectual centers of that time, opening himself up to the scientific community and making a name for himself at the international level.[28]

1.4 European Tour (1756–1759)

As mentioned before, from the middle of the 16th century until the beginning of the 18th century, a series of religious and political events took place which, combined with certain scientific developments, led to a change in mentality. These advances came to point out the possibility that man by himself using reason, without resorting

[26] Lambert's scientific diary was first edited by Karl Bopp in his Bopp (1915). Roger Jaquel, one of the great experts on Lambert, wrote in 1977 in connection with Lambert's collected works that «the most urgent and appreciated service would consist of providing a translation, or at least a new edition of Lambert's *Monatsbuch*» (Jaquel 1977, p. 95). One of Jaquel's wishes was already fulfilled in a recent publication (Bokhove et al. 2020). It is more than desirable that an English translation be done.

[27] Lambert (1755). An exhaustive list of Lambert's works can be consulted on the website *Johann Heinrich Lambert (1728–1777) Collected Works—Sämtliche Werke Online* written and designed by Maarten Bullynck: http://www.kuttaka.org/~JHL/Main.html. Also in the classic work by Max Steck *Bibliographia Lambertiana* (Steck 1970).

[28] The following section, and more generally the whole chapter, does not cover all of Lambert's journeys. For a more detailed map I forward the interested reader to Jaquel (1979, pp. 52–53). Jaquel explains (pp. 50–51), that this map is a corrected version of Max Steck's *Topologische Karte der Reisen von J. H. Lambert* (*Topologic map of J. H. Lambert's journeys*), 1951.

to revealed truth, could know how things work. Furthermore, this 180-degree turn —reason versus dogma— did not remain in the field of science but was generally extended to different aspects of life, changing all human activity.[29]

Although initially it may have seemed that this got religious thinking out of the way, this latter thinking was in fact a truly remarkable driver of this approach to knowledge. As new discoveries appeared, the argument that the order and mechanism that governs nature points to the existence of a creator was beginning to replace a priori reasoning and even the revelation of the Scriptures as the main proof of God's existence. To search for God, then, was to search in nature, so the implications for science were important (Fig. 1.3).

But this was a search based on observation and experiment, since «no logical argument alone could fathom God's free choice»,[30] so that the phrase «use reason» has now to be reinterpreted as evoking a way of knowing with a critical spirit and an open mind directed towards the natural sciences, and no longer as an a priori approach to the truth. A vision that also led to a shift in the field of mathematics from geometry to calculus, a tool that had been developed to respond to problems related to movement (mechanics). What we find in this century then, in what science refers to, is an eagerness to answer the questions posed by nature together with an experimental spirit,[31] wherein mathematics —mainly Calculus, a paradigm of reason and the correct method to follow—, played a central role. After Newton's marriage of experimentation and mathematics in his work, the debate centered on the proper balance between the two.

In late 1756 Lambert went to Göttingen, whose university was a good example of the shift towards the enlightened way of life. Since the Peace of Westphalia in 1648, the Wars of Religion in the Holy Roman Empire had ended, and the area enjoyed a healthy tolerance of religion. Scientific thought was steeped in Wolff's ideas: experimentation was used, but the approach was primarily rationalistic.

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[29] It is important to note, however, that this change was not immediate at all. In fact, up until mid-century the church was still gaining ground in some areas, and its dominance was strong in places such as France, Spain, Portugal and Hungary, making it difficult for innovative ideas to enter, whereas in other places they had already been introduced, such as in Great Britain, Holland or Prussia. Derek Beales in his chapter «Religion and culture» in Blanning (2000, pp. 131–177) makes it clear on page 133:

> Among all the titles generally bestowed on the eighteenth century, 'the Age of Religion' and 'the Christian Century' are missing. They should not be, because at any rate in the first of my periods the churches were in many respects still gaining ground. But this achievement has been concealed, because historians have greatly exaggerated the immediate impact on religion of two developments of the late seventeenth century, first, the supposed 'end of religious wars' and, secondly, the 'scientific revolution' leading to what Paul Hazard entitled 'the crisis of the European consciousness' and dated to the years 1680 to 1715.

[30] Hankins (1985, p. 3). This is typical of voluntarist theology, an important view since the Middle Ages (and one opposed to rationalist theology, cf. Leibniz) (I thank J. Ferreirós for the comment).

[31] In fact, Lambert had already built experimental instruments to carry out his own observations, a habit that he was to maintain throughout his life.

Fig. 1.3 Lambert and the «Colonne Lambert» erected in 1828 in front of both his house and the Saint-Etienne church in his honour in commemoration of the 100th anniversary of his birth (available at http://ark.bnf.fr/ark:/12148/cb419207773)

There, Lambert took law classes at the newly founded university and studied works by Bernoulli, who had developed Leibniz's Calculus to address mechanics questions —very much in vogue at that time, as for instance the brachistochrone—, like those of Euler, which, with the enormous influence that he was giving to Calculus, would lead to the formation of an increasingly autonomous discipline. In fact, only a year before Lambert's arrival in Göttingen, his famous *Institutiones calculi differentialis*[32] had come to light, one of the three treatises that would contain everything collected so far on the subject.[33]

[32] *Foundations of differential calculus*.

[33] The other two are: the two-volume treatise *Introductio in analysin infinitorum* (*Introduction to the Analysis of the Infinite*) in 1748, and the three-volume treatise which would arrive in 1768 *Institutiones calculi integralis* (*Foundations of integral calculus*).

Furthermore, he also knew two important academics: the astronomer Tobias Mayer,[34] who was in charge of the city's observatory and who years before had published the first map with coordinates of the Moon and some excellent lunar tables that would earn him an award from the Board of Longitude in London (posthumously);[35] and Abraham Gotthelf Kästner, professor of philosophy and mathematics at that university with whom he would correspond until his death. The influence that both Kästner and his student Klügel —whom Lambert also knew during his stay in Göttingen— exercised on Lambert would lead him to mark out the path that later generations followed towards the search for new geometries.[36]

It is also probable that Kästner, who shared Lambert's interest in the study of perspective, would have inspired him to delve into it. Both had published works on the subject years before, and at this time, and as recorded in his Monatsbuch, Lambert made different annotations until the publication of his main work on perspective in 1759, the one of September 1758 being especially interesting: «in Marseille I put down the foundations of perspective».[37]

Lambert's stay in Göttingen, where he was appointed corresponding member of the Academy of Sciences, lasted no more than a year. Political disputes in Europe due mainly to dynastic problems and territorial conflicts were abundant, and at the same time that he visited the city, specifically in the summer of 1757, it was invaded by France due to the Seven Years War. With no other option, he took his two pupils and made his way to the Netherlands, a neutral ground, and which together with Great Britian had been since the beginning of the century an example of religious tolerance and relative freedom, something that had promoted the new habits of the Enlightenment.

He spent most of the year during his stay in the Netherlands in Utrecht. It is recorded that while there he suffered a spectacular fall that almost costed him his life. He was unconscious for a whole day and, in fact, the doctor who treated him «advised him to abstain for a few years from serious reflections»[38] due to the severe

[34] Quite a few parallels are to be noted between Mayer's life and Lambert's: financial difficulties (he grew up in poverty) and the loss of his father in his youth. He was also a self-taught man when learning mathematics and physics. One of the first things he excelled at before dedicating himself to astronomy was cartography, a field in which later Lambert would also make important contributions.

[35] Some lunar tables were of great practical interest as they made it possible to calculate longitude offshore, but it was a difficult task since, unlike the planets that are attracted to the sun, the moon is attracted also to the earth. This exercised the minds of such great savants of the day as Euler, d'Alembert and Clairaut, who faced this «Three-Body Problem», one of the three tests that Newton's law of gravitation would have to pass for verification, together with the determination of the shape of the earth and the date of the return of Halley's comet.

[36] Given the fundamental importance of Lambert's investigations in this field, we will dedicate some lines to this issue in Appendix B.

[37] This Lambert's main work on perspective I have just referred to is Lambert (1759). The quote is taken from Andersen (2007, p. 638), who dedicates a chapter to Lambert's work since his contributions «are so outstanding in the history of perspective that they deserve a separate chapter» (in p. 599).

[38] Formey cited in Sheynin (2010, p. 142).

blow he had received on the head, althought with no success. From Utrecht, he made short trips to the main Dutch cities: Amsterdam first; the Hague later, where in 1758 he published his first book *Les propriétés remarquables de la route de la lumière par les airs et en général par plusieurs milieux réfringens, sphériques et concentriques*[39] about light paths in different mediums; and finally, Leyden where he met Pieter van Musschenbroek.

At the beginning of the century, Van Musschenbroek together with Willem Jacob's Gravesande and Hermann Boerhaave had published a series of works in which they followed Newtons footsteps by conducting experiments, which played a leading role (mathematics was necessary but subjugated to the experimentation, unlike what happened in Germany). Holland, led by the University of Leyden, became the primary follower of his method in Europe (along with Great Britain, naturally). Musschenbroek, who had to his credit, among other things, the discovery, independently, of the first type of electric capacitor (the one known as the Leyden bottle) met Lambert when he was already an older man. They seem to have had a conversation on different topics in which, contrary to his first impression, he was surprised by Lambert's extensive knowledge: «the roles of the interlocutors switched; Lambert became the teacher and Musschenbroek, the student».[40]
His last stop before returning to Chur was in France, where—despite the difficulties encountered by some of its most illustrious citizens such as Voltaire or Diderot—the central document of the Enlightenment, *L'Encyclopédie*, was released. Lambert arrived to Paris in the summer of 1758, and there he established a relationship with Charles Messier, an astronomer who was beginning to draw up his «Messier Catalogue» (still known and in use today), in which he classified certain celestial objects with the intent of carrying out an easier search for comets, his main objective, apparently motivated by his observation of the comet of 1744. But above all, he established contact with a man of already international fame who was part of the forefront of mathematics along with Euler and the Bernoullis, and who was to be an important agent in the change of mentality that was to occur above all in France from the second half of the century onward.

Lambert met d'Alembert in 1758 when he had left the Encyclopedia project, probably due to the differences that were beginning to emerge between its two directors. Diderot questioned mathematics as a tool, arguing that it alienated the scientist from reality, and d'Alembert, although he understood the need for observation, argued instead that mathematics should occupy a central position.[41] And it was not just a

[39] Lambert (1758).

[40] Formey in Sheynin (2010, p. 142).

[41] Criticisms of mathematics as a useless and obscure discipline were not new. Petrus Ramus had already in 1569 published *Scholarum Mathematicarum* in which he examined the reason for the small esteem in which scholars held mathematics. He (like d'Alembert later on) would point to the structure and methodology of Euclid's *Elements* —the classic text of mathematical instruction of the time— as the main problem (for more details see Schubring (2005, pp. 68–69) where the author claims this work by Ramus to be «the first methodological reflection on mathematics in print»). In relation to the dispute over the usefulness of mathematics around the figure of d'Alembert, see Richards (2006, pp. 702–704, 706).

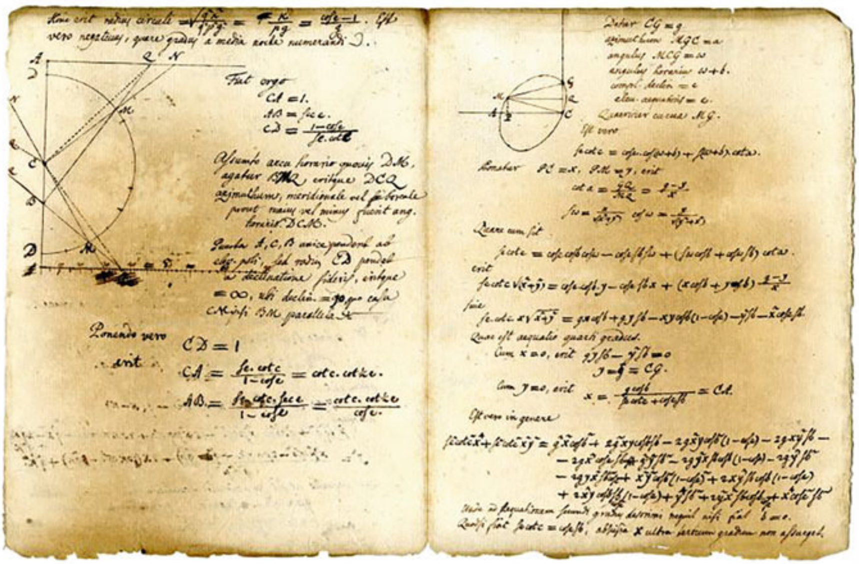

Fig. 1.4 Lambert's handwritten studies on Cartography, a discipline in which he would also leave his mark (courtesy of the *Archives de Mulhouse*)

few who subscribed to this view of mathematics as having little utility. Buffon came to call it empty and of little use to real life —products of the human mind and not abstractions (as d'Alembert saw them) of real objects that therefore benefit from his mathematical study. Diderot, who did not hide his joy when his colleague left the *L'Encyclopédie* —«the reign of mathematics is over»,[42] he wrote to Voltaire—, «did what he could to undercut the claims of d'Alembert's *"l'esprit géomètre"*».[43]

In any case, his «Discours Préliminaire»[44] for this same work and the first seven volumes of it that would appear between 1751 and 1759 left their mark, earning him great prestige. In fact, years before, and on two occasions, he had received offers from Frederick the Great to preside over the Berlin Academy of Sciences, proposals that he rejected, arguing that it would be audacious to pretend to be above Euler in a scientific institution.[45] It seems that unlike with Messier, with whom he would come to maintain a friendly relationship, the first impression of this illustrious thinker towards Lambert was not good, although after time he would end up recognizing his worth.[46]

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[42] Diderot quoted in Richards (2006, p. 706).

[43] Richards (2006, p. 706).

[44] *Preliminary Discourse*.

[45] Hormigón (1994, p. 27).

[46] These impressions will be discussed more closely later in this chapter.

This European trip that ended in the last days of 1758 served him both inasmuch as it allowed him to get to know in person the main lines of research of the moment together with some of the most relevant scholars in Europe, and to publicize his first works in the academic world. But now, as his service to the Salis family was reaching its end, he was faced with the problem of finding a stable position from which to continue his studies. His mind was on Göttingen, as is attested to by a letter of August 18, 1758 that he wrote while still in Paris and addresses to Albrecht von Haller, who had recommended Lambert while he was there, and who, together with Kästner, had helped him to disseminate his works (Fig. 1.4). Haller held the Chair of Medicine at the University of Göttingen and had been called by George II to inaugurate the presidency of the Academy of Sciences founded seven years earlier. After thanking him for his help, Lambert asked him to intercede for him:

> My service in the de Salis family will end before October and I ought to regret [the imminent loss of] the free time they had been willingly leaving me for working on such subjects. I do not know when will I be able to resume working on them[47] [...] I sincerely admit, Sir, that I expect to find it again in Göttingen and nothing would have pleased me more than an invitation to the chair of philosophy. I understand well enough that when competing for that position it is easy to become an instructor waiting for a vacant chair, and I know not less that the prime minister von Münchhausen much prefers literature to provide means for those, who, armed with a recommendation, ask to teach. I know very well, however, what does it mean to give lessons for a living and how much time necessary for working at the furthering of sciences is lost thereby. You know it, Sir, and your example vividly proves that the glory of a university much less depends on those who are only teaching than on those who in addition acquired reputation by their writings. I do not deny that it is that glory to which I aspire, and I do not wish anything so much as achieving successful development [in science] [...] How satisfied will I be if your recommendations will assure me such a possibility or if the actual circumstances at Göttingen University will permit an invitation which can benefit me.[48]

Despite the fact that Haller had supported Lambert in his time in Göttingen, his attempt at getting a position at the university came to nothing. He left Paris to return to von Salis' home in Chur, via Marseille, Nice, Turin and Milan. Finally quitting the service of the von Salis family in May 1759, Lambert would lead, for the next five years, a restless life oriented towards a safe position from which to continue his investigations, and towards finding possible editors for his works.

1.5 Itinerant Period (1759–1765)

After a short rest at the home of his former pupils, Lambert departed towards Mülhausen with the desire to spend some time with his family. Before going to Mülhausen, however, he stoped off in Zürich where he was elected a member of the Society of Physics after being enthusiastically received by, among others, Johannes

[47] He refers to Lambert (1779) and Lambert (1760), both of which will be published later (the first of them will be published posthumously).

[48] Lambert quoted by Wolf in Sheynin (2010, p. 153).

Gessner, physician and botanist with whom he made astronomical observations. In the weeks in which he stayed here, he published his *Die freye Perspektive*, a treatise in which he studied how to draw accurately in perspective. Once in Mülhausen, he spent three months with his mother (his father had died in 1747), and from there he headed towards Augsburg. After this, the mother and son were never to see one another again.

In Augsburg, he met the famous maker of scientific instruments Georg Friedrich Brander with whom he was to maintain a twenty-year correspondence, and participated as a founding member in 1759 at the creation of the Bavarian Electoral Academy of Sciences.[49] The agreement that they came to was that in exchange for sending their notes and assisting them in general with their advice, he would receive the title of honorary professor and a pension of 800 florins, and the freedom to settle beyond the limits of Bavaria, a rare type of flexibility which shows the status and reputation that Lambert already had as a scientist. This would allow him to proceed with his investigations in a relatively safe and stable manner, fixing his attention on definitively finishing the jobs he had described to Haller in his letter.

The first of these was *Photometria sive de mensure et gradibus luminis colorum et umbra*,[50] a work that he finished in 1760 which dealt with the behavior of light when passing through different mediums (for example plaster or paper), independently reaching results established years ago by Pierre Bouguer, a true pioneer on the subject, and expanding his research. One year later, motivated by his encounter with the great comet of 1744, he published a small work, *Insigniores orbitae cometarum proprietates*,[51] in which his famous theorem on elliptical orbits made its first appearance, a theorem that would later be applied by Olbers to calculate the orbits of the comets.[52]

But what is more interesting, perhaps, is *Cosmologische Briefe über die Einrichtung des Weltbaues*, which he published in the same year,[53] and which shows a Lambert who broke with the dominant current of the Enlightenment to advocate an idea more typical of Romanticism. This is clearly seen in his organicist vision of the universe. Viewing the universe as a whole —similar to the functioning of an organism— clashed with the prevailing mechanistic conception. But it is also possible to grasp his vision by paying attention to how he justified its stability: this is a work done by God according to His intentions; creating it so that it would later be destroyed would not make sense. Therefore, the end of the universe is the preservation of life, hence its stability. The appeal of the theological argument shows another clear

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[49] The *Churfürstliche Akademie der Wissenschaften* (Sheynin 2010, p. 149 note 9).

[50] Lambert (1760). This work is usually referred as *Photometry*.

[51] Lambert (1761a).

[52] Olbers' method is still used today.

[53] Lambert (1761b).

Fig. 1.5 Portrait of J. H. Lambert by Frédéric-Emile Simon circa 1836 (available at http://ark.bnf.fr/ark:/12148/cb419202991)

difference with regard to the mainstream ideas,[54] which demanded that everything be explained in terms of movement and not final causes.[55]

In this work Lambert tried to provide answers to questions bearing upon the nature of the Milky Way —a topic often approached during this century and in which the theoretical contributions of Immanuel Kant, Thomas Wright, and Swedenborg stand out— and proposed a hierarchical cosmos: a sun with planets forms first-order systems; a sun with a few million stars forms second-order systems; and groupings of the latter would form third-order systems, which would include the Milky Way (he even speculates on the existence of possible fourth-order systems —galaxy clusters— forming the universe). Let us mention that the real problem involved in the study of the universe that had led Kant to affirm that cosmology could not constitute a science in the strict sense was that direct empirical contact was lacking. To be able to answer certain questions it was necessary to let speculation and imagination play a role but without losing sight of the experience. Lambert was therefore incurring a risk he was aware of, although his own words appear to reflect not so much concern:

> I can consider my conclusions as a very daring model, especially because I live in times in which the freedom to order nature according to one's own criteria has been banished altogether. And I do not order only isolated parts but all of nature and the entire environment of creation according to my criteria. Could I be any more impudent?[56]

The impact of his ideas on some of the scientists of the time was enormous, increasing his status and further contributing to his reputation as a genius (Fig. 1.5).[57]

That same year, after his latest publications, he undertook numerous trips. He visited the University of Erlangen and then went to Chur and Zürich —where he was elected an honorary member of the Physical Society «as a person whose penetrating mind reveals the truths in the most difficult sciences, discovers new truths and exposes secrets»—,[58] before returning to Chur in the summer of 1762, where he would remain until the autumn. From here, he went to Italy, and in December 1763, he arrived in Leipzig where he focused on his philosophical work that would finally be published in 1764 in two parts under the title *Neues Organon*.[59]

It was also at this time that the economic support that allowed him to act with a certain calm started to dry up. As a result of different discrepancies that arose between

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[54] In any case, although mechanism versus teleology was certainly the general trend (for example, in France they were very common ideas), it was not the only possibility. In Germany there was enormous influence from the ideas of Wolff, learned by Lambert in his first readings, and through this those of Leibniz, who argued that mechanism and teleology were not only not at odds, but that the source of mechanics were the final causes.

[55] Hankins (1985).

[56] Martín et al. (2007, p. 303).

[57] In letters reproduced in Sheynin (2010), comments can be found in relation to this work such as: «Lambert is Newton's interpreter and rival» (p. 157) or « Lambert, one of the most astonishing geniuses of the 18th century» (p. 158). It is true, however, that some of those who flatter him have no qualms about criticizing him due to the lack of clarity that he shows in his writings (anyway there are divergent opinions).

[58] Cited by Wolf in Sheynin (2010, p. 159).

[59] Lambert (1764). Lambert's philosophical work had an enormous impact, especially in Germany.

the Bavarian Academy, which reproached him for not taking sufficient account of their interests, and Lambert, who reproached them for having neglected his advice, the relations between the two came to an end, cutting off his pension as a member. What he wrote to Euler by letter points in this direction:[60]

> It is also in an unexpected way for me and without making use of my observations about Mpt. [?] that they have published the observation of ♀ in ⊙ that the academics had made in Munich.[61] Of the six observations, they did not publish more than two, those that seem to match better, and I have had both to see this work printed in a journal, and that the editors had no scruples to treat the conclusions as suspicious. It seems that the Academy believes that it sufficed for it to handle this just by itself, since otherwise I would have been more included in these processes, and would, needless to say, never have allowed these things to occur.

This put him in a delicate situation that he apparently did not care too much about, for he later rejected an offer from the Saint Petersburg Academy. Actually, Lambert's interests were in Prussia, specifically in Berlin and its Academy of Sciences, where he hoped to achieve the desired stability with the help of Sulzer and Euler.

1.6 Stability. Lambert and the Berlin Academy of Science (1765–1777)

Since the beginning of the century, numerous societies and academies had begun to emerge in different parts of Europe which, with the support of intellectuals and leaders more or less integrated into the enlightened movement, took study and critical research as their main objective; several have already been cited in this chapter.[62] The role they played was important, first in terms of science, since universities of the time were reluctant to its teaching and research,[63] and secondly, in reference to the scientists, since they provided a stability difficult to achieve another way.

One of these new academies that would end up having the most relevance in the 18th century was the Berlin Academy of Science, although its beginning was

[60] As of July 12, 1762 Bopp (1924, p. 28). Thiébault II (1806a, p. 291) comes to say that Lambert decided to leave due to problems that had gotten him some envious rivals in the Academy. Euler seemed to be of the opinion that Lambert's relation to the institution had worsened due to religious differences between the Protestant Swiss and the Jesuits Sheynin (2010, p. 172 note 22). The reference in Calinger (2016, p. 563 note 51) —«A Protestant, he [Lambert] could not work with the Bavarian Jesuits»— seems excessive.

[61] The symbols represent Mercury and the Sun respectively.

[62] They were modelled after the *Royal Society of London* (1662) and the *Académie des Sciences de Paris* (Parisian Academy of Sciences) (1666).

[63] The first to make room for other types of subjects apart from the classics that were already taught in the universities were the Pietists, who in their important educational reforms introduced novelties such as the teaching of geometry and mechanics (Blanning 2000, p. 136).

fraught with difficulties.[64] The Academy was founded by the first King of Prussia, Frederick I of Prussia, and III of Brandenburg (1657–1716), known as «the Elector», thanks to the advice of Leibniz on July 11, 1700, coinciding with the King's birthday. However, the real (and royal) engine behind the establishment of the Academy was his second wife, Princess Sophia Charlotte of the house of Hannover, who had financially supported Leibniz in order for him to undertake this project. With difficulties from the beginning due to financing problems, it was not inaugurated until 1711. Leibniz, who died in 1716, never saw how the institution achieved a prominent place among the academies and societies that, at that time, were dominated by those of London, Paris, and St. Petersburg.

At first, the Academy, which had been founded under the name *Societas Regia Scientiarum*, issued periodic publications in its journal *Miscellanea Berolinensia*, to disseminate —in the same way as their European counterparts— the research carried out by its members. Research was framed within the four classes that the Academy was divided into in its first stage:

> physics, medicine, and chemistry; mathematics, astronomy, and mechanics; German language and history; and literature, stressing oriental writings for missionaries spreading the Gospel to the east.[65]

Within this standard, the first activities of the institution barely had any impact, and under the mandate of his son and second King of Prussia, Frederick William I of Prussia (1688–1740) known as «The Soldier-King» —a man more concerned with military duties than with academic matters— the newly created Academy not only did not improve, but nearly disappeared. It suffices to show the little respect he had towards these matters that «in 1731 he named the royal jester at court its vice president».[66] But from 1740, the year in which Frederick «The Great» (1712–1786) ascended to the throne, things, although slowly, began to take a different turn.

Before even becoming King around 1736, his correspondence with Voltaire showed the future monarch's concern for the poor state of the old Academy, emphasizing his interest in making it a worthy competitor on the European scene. The choice of the president to carry out this renovation was of vital importance —of course from an academic point of view but also from an advertising point of view— and apparently it was Voltaire who was the first choice of Frederick, but after he refused, all efforts were directed towards Maupertuis, a man who in his time «enjoyed the highest celebrity».[67] Maupertuis would end up accepting the presidency of the Academy in 1746, but initially —he arrived in Berlín in 1740— the King's plans to revitalize the neglected institution were relegated to the back burner for his foray into the Silesian War.

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[64] As for the Berlin Academy of Science, the sources consulted are: Thiébault I (1806b), Thiébault II (1806a), Cajori (1927), Aarsleff (1989), Begehr et al. (1998) and Calinger (2016) (especially, but not only, chapter 6), as well as some correspondence that will be cited in due course.

[65] Calinger (2016, p. 177).

[66] Calinger (2016, p. 177).

[67] Thiébault II (1806a, p. 282).

Among those who also demanded the monarch's prompt attention was Euler, who after negotiating the conditions of his contract, had accepted the offer proposed by Frederick via his friend and ambassador in St. Petersburg, Ulrich Friedrich von Suhm, only fifteen days after his ascent to the throne:

> Do whatever you can to hire Mr. Euler, the grand algebraist, and bring him with you if you can. I will pay him a salary of ten or twelve escudos.[68]

Euler's salary which had finally been raised up to 1600 escudos to match what was offered in Russia, gives an idea of the importance that he had for the King —compared with the average salary of the junior members of the Academy, which was about 300 thalers[69]

Euler, who arrived in Berlin on July 25, 1741, eager to get to work at the new Academy, was also facing an uninspiring scenario. The next couple of years the King was too busy with the war to focus on other issues, so, with much insisting and with the help of French intellectuals from the city and court nobles, Euler gave life to the *Nouvelle Société littéraire* (1743) —a kind of intermediate step between the old *Societas* and what would become the new Academy—, that would meet weekly to report on the results obtained by its members. Classified into three classes —mathematics, literature, and physics— it would be divided into 20 ordinary members and 16 honorary or aristocratic members. Half of them were Huguenots, and the official language would be French, which shows the strong presence of French in the city around this time,[70] the strong influence of the French Enlightenment, and the King's own Frenchified tastes.[71] The idea was for the *Nouvelle Société* to become the new and definitive Society, but it eventually merged with the old, creating the *Académie Royale des Sciences et Belles-Lettres de Prusse*. The founding statutes were approved taking advantage of the 32nd birthday of the King, January 23, 1744.

Anyway, it was not until March 3, 1746 that the Academy officially made Maupertuis president, and it was not until June 2nd of the same year that they even read the constitution of the new Academy. At the initiative of the new president was mandatory attendance for all members at plenary sessions, which were to be held every Thursday regardless of the class to which they belonged, a novelty compared to other academies which gave their members a broader knowledge of the main lines of research being carried out in other fields of study. In these sessions, in addition to

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[68] As of June 14, 1740 Preuss (1850, p. 391).

[69] Calinger (2016, p. 170). As can be seen, in some cases French escudos are mentioned —for example, in this letter or in another from Lagrange, which will be cited later— and in others, the thalers («reichsthaler») as in the case of Calinger, which is presumably the currency used for payment of salaries. Anyway, keep in mind that the equivalence at that time was 1 escudo = 1 thaler Kindleberger (2006, p. 475).

[70] See Calinger (2016, pp. 195, 196). Aarsleff (1989, p. 194) gives the data that around 1740, 20% of the population were Huguenots, a high percentage of whom had probably escaped from France by reason of religious persecutions —as commented at the beginning of the chapter— and had been welcomed by Frederick William I of Brandenburg, «the Grand Elector» (1620–1688), himself a Calvinist.

[71] His own education had been carried out by Huguenots Aarsleff (1989, p. 194).

giving an account of different activities, the members had to read their works, being allowed to use either Latin, German or French to do so. In any case, as Maupertius wrote:

> French has been substituted for Latin in order to ensure a wider readership for the *Mëmoires*, for the knowledge of Latin is clearly declining while the French language today is almost in the same situation as Greek at the time of Cicero; it is taught everywhere and people eagerly seek books written in French.[72]

And so, since the internationalization of the Academy was the main objective, it was decreed that the works included in the *Memoires* were to be written in the vernacular language.

The Academy was divided into four classes, a slightly different division than that of the *Nouvelle Société littéraire* three years later. Three of them —the first, the second, and the fourth— were based on topics also dealt with in other academies, but the third class was homegrown and more focused on the initiative of the King. In the aforementioned plenary session of June 2, 1746, the division was established in the following terms:[73]

1. The Experimental Philosophy Class will include Chemistry, Anatomy, Botany and all the sciences that are based on experience.
2. The Mathematics Class will include Geometry, Algebra, Mechanics, Astronomy and all the sciences whose object is abstract study, or Numbers.
3. The Speculative Philosophy Class will be applied to Logic, Metaphysics and Moral.
4. The Belles Lettres Class will include Antiquities, History and Languages.

In that same session certain points were specified that Dieudonné Thiébault, himself member of the Academy since April 5, 1765, synthesized in Thiébault II (1806a, p. 283):

> All theological and political discussion was excluded from the academy. Each class had a director, who was chosen from its own body, and being composed of six resident members, the number of ordinary academicians extended to twenty-four, besides the perpetual secretary and the president.

In order to defray the expenses incurred by the Academy in the form of accommodation, buildings, or salaries, Thiébault went on to say that:

> the king assigned to his academy, besides the necessary buidings and lands, first, some extensive plantations of mulberry tress, from which great expectations were formed, but which in the sequel proved of small value; secondly, the exclusive privilege of publishing the king's edicts and the geographical charts, which were scarcely more productive than the former; thirdly, the exclusive privilege of composing and publishing almanacks, an article which, however insignificant in appearance, is the principal source of the wealth of the academy.[74]

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[72] Quoted in Aarsleff (1989, p. 196).

[73] The transcript of this plenary session can be seen on the website of the *Archive of the Berlin-Brandenburg Academy of Sciences and Humanities*.

[74] Thiébault II (1806a, pp. 284–285).

The four classes mentioned above would be a benchmark in Europe at the time. Their members would do first-rate research, and its class of speculative philosophy —one of a kind in Europa and the pride of the Academy— would bring the Academy international recognition.[75] In any case, it should not be forgotten that, despite Frederick's clear animosity towards mathematics,[76] much of the Academy's fame and high esteem would be thanks to having, among its other members, four of the most outstanding mathematicians of the time: Euler, head of the mathematics class; Lagrange, Euler's substitute after his move to Russia; Johann Bernoulli III, called by the King in 1764 to reorganize the astronomical observatory; and Lambert.[77]

Lambert, with his eye on the Berlin Academy, tried to obtain the favor of the King through Sulzer and Euler. Sulzer, who served as director of the philosophy department, had invited him on numerous occasions to Berlin, so he threw himself into it. Despite their achieving no progress at first,[78] when Lambert arrived in Berlin, they decided to try again. Sulzer was called to Potsdam where Frederick had his summer home, and there he began passionately speaking about the candidate, whom he deeply admired, to the people close to the King, with the intention of reaching the King's ear. As a result, Sulzer returned to Berlin to find a letter urging him to send Lambert to Potsdam the next day for an interview. Aware of Lambert's strange behavior and fearful that an audience with the King could go wrong, people close to Lambert objected, saying that their luggage has not yet arrived. The King simply replied that he wanted to meet the man, not his clothes. The last attempt to avoid the meeting was to warn the King that Lambert could not present himself properly, to which the King replied that they brought Lambert at night and with the light off. Finally, the interview occured:[79]

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[75] Aarsleff (1989, p. 198).

[76] In this regard see Cajori (1927).

[77] Begehr et al. (1998, pp. 1, 6).

[78] Calinger (2016, p. 428) writes:

> In April 1761 Euler nominated Lambert to be a regular member of the Royal Academy of Sciences in Berlin, and he was unanimously elected. But late during the war Frederick, working to reestablish his control over the academy, withheld approval.

[79] This anecdote is reproduced in almost every biography of Lambert, so it would be interesting to see if it is possible to know of its authenticity. The closest you can get to proving it is by making use of the stories of those who had heard the opinions of the King at dinner that same day, in which he complained about Lambert's mannerisms, or from some other member of the Academy who had heard something about the meeting. The version given here is the one reported by Dieudonne Thiébault in Thiébault II (1806a, p. 293), who by the way in Thiébault I (1806b, p. vi) says:

> The first law which I prescribed to myself on entering upon this work, and from which I have never deviated even in thought, was to write with the strictest fidelity respecting the facts it should contain. I solemnly declare, no single word appears in it that has not my entire belief. Some readers will perhaps oppose to this assertion, the particular conversations which I have put into the mouths of the greater part of the persons who figure in my scene, such as Frederick, MariaTheresa, &c. As to this I can affirm that I have not only ascribed to my speakers no thoughts which were not really their own; but I can further take upon me to

F: *Good day, Sir. Do me the favour to inform me which of the sciences you have particularly studied?*
L: *All of them, Sire.*
F: *You are, then, a skilful mathematican?*
L: *Yes, Sire.*
F: *Under what professor have you studied the science of mathematics?*
L: *I was my own instructor, Sire.*
F: *You are then a second Pascal?*
L: *Yes, Sire.*

Although to a naive Lambert the impression left by the meeting was positive, however, what mattered was Frederick's impression. Frederick was unaccustomed to such arrogant language, and therefore had been left with a rather bad impression. Sulzer then endeavored via letter to make the King's people see that they should not focus so much on the little things, but rather on the enormous knowledge possessed by Lambert, and that since he was once again receiving offers from Russia, they ran the risk of losing the opportunity to add into their ranks a man of great worth.[80] Frederick heeded the wise advise he was given, and, even though «M. Lambert was no less worthy of filling a place in the class of mathematiks or speculative philosophy, than in that of natural philosophy; this is proved by his works»,[81] Lambert was finally included as part of the Experimental Physics section on January 10, 1765. In any case, the King made it clear in a letter to d'Alembert the impression produced by the new tenant of the Academy:

> I have practically been forced, so to speak, to take the surliest creature alive in the universe and include him in our Academy. His name is Lambert, and although I can verify that he does not possess common sense, it is said that he is one of the greatest geometers of Europe. But, since this man ignores the tongues of mortals and does not speak more than in equations and Algebra, I do not have any intention in the short term to have the honor of conversing with him. On the contrary, I am very happy with M. Toussaint, whom I have procured. His science is more human than the other's. Toussaint is an inhabitant of Athens, and Lambert is a Caribbean or a savage from the Kaffir's coast. However, the entire Academy kneels before him, even Euler, and thus this animal, totally muddied with the dirtiest pedantry, receives these homages much like how Caligula collected from the Romans when trying to pass as God. I beg you not to let these little anecdotes from our Academy out of your hands.[82]

The image that the members of the Academy had of Lambert as one of the greatest geometers of Europe served for Frederick to choose him as a new member, but did not sweeten his opinion about him. The answer given by d'Alembert to the King

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> declare, that the very turn and way of presenting the thought is genuine and not of my own invention.

In Sheynin (2010, pp. 144, 172 note 30) other versions of the same interview can be read, the first of them by Formey in his *Eulogy on Lambert* —rather abbreviated— and the second by Graf, which is practically identical.

[80] See Cajori (1927, p. 127).

[81] Thiébault II (1806a, p. 290).

[82] It is not dated but is probably from January 1765 (Lalanne 1882, p. 142 note 1).

made it clear that if it had been up to him, Lambert would not have been elected a new member:

> I am only acquainted with one work by M. Lambert, which is good, but does not appear to me comparable to any of the works of Euler; and, if the latter be on his knees before M. Lambert, as your majesty has done me the honour to inform me he is, we must say of M. Euler as has been said of La Fontaine, that he was filly enough to believe Aesop and Phaedrus had more wit than himfelf. Not that I mean to derogate from the merit of M. Lambert, which must be very substantial, since it is so adjudged to be by the whole academy: but there is more than one honourable niche in the temple of the sciences; according as there is, if we believe the gospel, several mansions in the house of the celestial father. M. Lambert perhaps is exceedingly worthy of filling one of those niches. I am beside informed he has written several excellent works, which I have never read. I should think him tolerably well provided for when he should be, to speak mathematically, in the same ratio to Euler as Descartes and Newton are to Bayle, according to your majesty; or as Bayle is to Descartes and Newton, according to a mathematician of your acquaintance; or, again, to use a comparison which is not subject to contradiction, in the same proportion as Marcus Aurelius and Gustavus Adolphus are to a monarch whom I dare not name.[83]

It seems clear that the King's not having asked d'Alembert for advice before incorporating Lambert into the ranks of the Academy was a stroke of luck for him —contrary to the suggestion of Aarsleff (1989, p. 204)—,[84] since his influence on Frederick in the hiring of new members was huge, acting (via letter) from Paris almost like a president in the shadows,[85] an attitude that some of the academicians disliked.[86]

D'Alembert, who in the previous letter corrected the King by exchanging roles between the Descartes-Newton duo and Bayle based on their (most authoritative) criteria, would receive in the following years first-hand accounts about the high esteem that members of the Academy had for Lambert. In fact in 1769, he asked Lagrange —who had replaced Euler as head of the mathematics class after his departure in 1766 thanks to the advice of d'Alembert— for references about him:

> Regarding your Academy, I always forget to ask what you think of M. Lambert; what I have read about him so far does not seem to me the greatest relevance: it is said, however, that Euler holds him in high esteem.[87]

Note that in 1769, Lambert had already published works of great quality, among them being the one that has concerned us in this book. This fact, backed up by highly positive opinions expressed by some of his most illustrious contemporaries —Euler's case being rather remarkable— shows that this lack of recognition was probably based on a lack of knowledge on his work.[88] Lagrange's response —another giant of the 18th century— in July of that same year, underpins this idea:

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[83] As of March 1, 1765 (Holcroft 1789a, Vol. 11, pp. 20, 21).

[84] «It was probably also on d'Alembert's recommendation to the king that Johann Heinrich Lambert was nominated».

[85] In fact, if he was not president, it was because he did not want to be, since after the death of Maupertuis in 1759, the King had offered and asked for him to be, both actively and passively.

[86] Such is the case of Euler who found it unacceptable (Calinger 2016, p. 431).

[87] As of June 16, 1769 (Lalanne 1882, p. 135).

[88] In any case, it must be kept in mind that a large part of Lambert's works are written in German, which most likely diminished its reach and diffusion.

> M. Lambert, on whom you wish to know my opinion, is undoubtedly one of the best individuals of our Academy; a hard worker who practically holds up alone our Class of Physics. He dominates analysis, but his strength is Physics about which he has provided an esteemed Work, entitled *Photometria*, that is to say on the measure of light; there is above all an excellent memoir of him on magnets in Volume 1766.[89]

The high esteem in which Lambert was held by his colleagues at the Academy was therefore clear, although he was not only known and recognized for his knowledge but also for his peculiar behavior. Lagrange, who was not, at the beginning, left indifferent by Lambert either, continued by saying:

> For the rest, there is something singular in his bearing and in his conversation that is displeasing at first, and I am not surprised that the king did not like him, having I myself had a hard time adjusting to his manners. He was, or at least found him to be, so proud of himself when I arrived here, that I made the decision not to frequent him, but at the same time not to miss any opportunity to underrate him; this has made it easier to cope with him, and today we are quite good friends. He is not receiving more than 500 escudos from the Academy, and, if you have an opportunity to procure him a raise, I assure you that you woud do a good deed, for he is certainly one of the members to whom our Academy owes the most.

In his discourse of reception on January 24, 1765, *Discours de réception de M. Lambert comme membre de l'académie*,[90] Lambert made it clear that his primary task was to address the theory of fire and heat in a more complete and systematic way than before through his *Pyrometrie*, «A vast and complicated occupation as never before!».[91] But he also dedicated his discourse to defending his vision of science as a whole, explaining how in this way, the different branches of knowledge are interconnected. From the connections between mathematics and physics —a subject of debate, as already noted, throughout the century, and one that he balanced in a synthesis between empiricism and rationalism, the basis of his methodology and philosophy—, to the relationship between science and poetry, or between physics and history. In relation to this discourse, Thiébault related an exquisite anecdote, which despite being long, certainly deserves a place in this biography, for it clearly shows, as he himself said, «the simplicity, artlessness and frankness of M. Lambert's character»:[92]

> The new academician was now employed in composing his inaugural discourse, and determined to resolve in it a question of importance respecting the reflection of light. He had still, however, to this effect some experiments to verify, for which he stood in need of a large looking glass, while his whole stock of furniture afforded only a small pocket-glass barely large enough to allow of his adjusting his wig in it. The best remedy he could think of was to go into the principal coffee-house of Berlin, situated opposite to the castle. On entering one of the rooms on the first floor, he bowed in his accustomed manner, without looking at them, and throwing his head diagonally from one side to the other to some officers and other persons of the town, who were playing at tarocs,[93] passed on to a large mirror which

[89] As of July 15, 1769 (Lalanne 1882, p. 141).

[90] *Discourse of reception by Mr. Lambert as a member of the Academy*.

[91] Lambert (1765/1767, p. 215). This work will only appear after his death.

[92] Both this quote and the one to come, in Thiébault II (1806a, pp. 294–295).

[93] «A sort of playing cards, but marked differently from the common ones» (note by Thiébault).

> happened to be placed in the lightest parto of the room; he then drew his sword, aimed it as if against an adversary, drew back, advanced, in short, threw himself into the different attitudes of a real encounter, at the same time profoundly meditating on what he saw and did. He pursued his experiments for the space of half an hour, without the least consciousness that the spectators, who knew neither his person nor what to think of the exhibition they had witnessed, had concluded he was a lunatik, and were actually holding themselves ready to seize and disarm him should it be necessary.
> When M. Lambert had ended his experiments and his reflections, he put his sword quietly into its scabbard, cast a look of indifference on those who surrounded him, bowed to them in the same manner as when he entered, and returned home to compose a memoir worthy the admiration of the learned.

The twelve years that he stayed at the Academy were the best of his life, although at first —as should have been clear— his peculiar personality complicated his relationships with his colleagues. Thiébault II (1806a, pp. 296–297) told how, on one occasion, he asked Lambert about the list of the most celebrated geometers according to his estimation. Lambert placed Euler and d'Alembert in the same rank considering them «as it were but one person», and argued:

> not because their qualities are similar in every respect, but because each has eminent qualities that compensate those deficient in the other. M. Euler has more simplicity and promptitude, perhaps even a greater abundance, than M. d'Alembert; M. d'Alembert has more subtlety, sagacity, and elegance, than M. Euler. In profoundness of understanding and fertility of invention they are equal. It is impossible to give the preference to either.

Continuing with his ranking, Lambert said:

> M. de la Grange is at present the second: I say at present, because there is every reason to believe that he will not long remain inferiour to the first. The third is myself. I can proceed no further in this classification, because I know no other geometrician worthy of being named.

Apparently, a young mathematics teacher who was teaching artillery students not only disagreed with him on this last detail, but also positioned himself as third:

> No sooner had he pronounced the words than M. Lambert advanced with a resolute step before him, as though he would have barred the way, and then looking him stedfastly in the face, he burst into a fit of laughter and turn on his heel.

It is perhaps because of details like this that Johann Bernoulli III, one of his best friends, and, in fact, one of those who knew him best, wrote this about him:

> Lambert casts a shadow on his great merits by unimaginable conceit. Partly he caused us to lose Euler,[94] and among his colleagues he is only getting along with me. I do not quarrel

[94] Sheynin opens a note here (included in Sheynin (2010, p. 173)) to try to throw a little light on this. The idea that can be extracted by reading the literature about it —without pretending to be better informed than Bernoulli himself— is that Lambert could have been just one more drop in a glass almost filled by a burned-out Euler. No matter how good a mathematician he was, for the king, Euler would never measure up to the like of d'Alembert or Voltaire, men with class and polished conversation. Furthermore, when Maupertuis began spending long stays outside of Berlin near the end of his life, and also since his death, Euler acted as president of the Academy, but he was never named as such despite d'Alembert's recommendations, something that could have been felt to be a lack of recognition of their hard, brilliant and voluminous work. The king apparently

> with him although we had been taking meals together all the time he lived in my place. His conversation on all the sciences is instructive. If you do not ask him about anything except his own ideas, and do not interrupt or contradict him, he will speak for three hours as though reading from a book.[95]

In any case, his talent was beyond doubt, and over the years they would end up convincing themselves that his behavior was due more to excessive naivety and a deficiency in his ability to interact socially with normality than to a bad temperament. Thiébault himself, who agreed with what Johann Bernoulli III said about what seems to be something quite characteristic of Lambert's personality, showed signs of this admiration:

> When I happened to meet him in company, or in my walks, my first care was to propose to him some question that interested me; for when once entered into a discussion, on whatever subject it might be, it was not longer possible either to stop or interrupt him. He never failed, from the first moment, to take so clear and comprehensive a view of his plan, and adhered to it so closely, that to divert his attention was impracticable. The order of his ideas was always regular and perfect; if objections were proposed to him, he paused no longer than was necessary to hear them to their end; he never, however, answered them, but resumed the thread of his argument as though he had not been interrupted, because the objections he had heard would, he perceived, occur at a different time and more seasonable order, and that it could not but be disadvantageous to the discussion to deviate from the principle he had first laid down. I have a hundred times put him to the trial in this respect, and found him always the same. He was truly a machine to grind dissertations, but a perfect machine.[96]

He even ended up winning the admiration of the King, who, in 1770, would raise his salary and put him at the head of the economic commission, to which he already belonged. Apparently, Lagrange's request to d'Alembert to intercede for him worked, and in 1769, d'Alembert mentioned that in his last letter to the King «I have also said a few words about M. Lambert, according to how well you spoke about him»,[97] words he addressed to the King that same day:

> The *Memoirs* from your Academy of Sciences are excellent works and demonstrate that it is one of the best composed Societies of scholars in Europe. I am not speaking only of M. de la Grange, whose merit is well known to Your Majesty; I speak, among others, of M. Lambert and Beguelin, who contribute excellent Memoirs to this Collection and which seem to me worthy of the kindness with which Your Majesty has always honored merit.[98]

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distrusted the Swiss's administrative skills, and should have blamed him for the high expenses of the Academy during war. After he ordered a commission to investigate why revenues had fallen so low —a commission led by Euler himself and in which Lambert was among its members— the tensions between them got worse. Lambert could have been a drop more, though (as pointed out by Sheynin in his note) would have been so without in the least intending to be. Calinger (2016, p. 442) says that «the idea that Lambert worked to drive Euler from Berlin is erroneous».

[95] As of October 11, 1766. Quoted by Wolf in Sheynin (2010, p. 164).

[96] Thiébault II (1806a, pp. 295, 296).

[97] As of August 7, 1769 (Lalanne 1882, p. 147).

[98] As of August 7, 1769 (Lalanne 1882, p. 147 note 1).

Frederick II's response showed signs of the change in his perception:

> The three individuals you talk to me about are, without a doubt, the best we have in this corps.[99]

With a frenetic pace of study, Lambert devoted himself completely to his research, producing more than 150 works for publication in the most diverse areas. Indeed, although he focused mostly on scientific subjects, he continued to write also about philosophy and was the only member in the Academy's history to publish memoirs in all four divisions of its research journal. Proof of this versatility and once again of the King's change of perspective and esteem towards Lambert can be seen when d'Alembert raised a subject to be adressed in the speculative philosophy course; the King immediately thought of him:

> You speak of a question to propose to the academy. Alas! We have recently lost poor Lambert, one of our best members. I know not who could treat the subject philosophically.[100]

In 1775 a cold that he had recklessly neglected became more severe during the winter. Despite the advice of his friends, he refused to take it seriously or get treatment from a doctor until a very late stage, applying his own remedies instead, thus worsening the situation.[101] His colleagues last saw him «at the Assembly of 18 September, more dead than alive, and he even experienced convulsive symptoms which frightened those who noticed them».[102] The polymath genius died of tuberculosis on September 25, 1777, in Berlin at the young age of 49, leaving behind a vast scientific legacy. To serve as a conclusion, these sad words by a dejected Lagrange in a letter to d'Alembert:

> I am so saddened by the death of my colleague M. Lambert; it is an irreparable loss for our Academy and for Germany as a whole; he eminently possessed the rare talent of applying calculus to experiments and observations, and of extracting, so to speak, everything that could be regular. His *Photométrie*, a Work little known in France and even in Germany, is a true model of this type of research; he was also versed in calculus and was not unaware of a single one of the different branches of Analysis and Mechanics. The three Volumes of Memoirs that he produced in German, some years ago, contained excellent things, and it would be desirable if someone wanted to translate them. There is in all his research a great sharpness, and he possessed, above all, the art of achieving the simplest results, even in the subjects that seemed most complicated. He allowed himself to die little by little of

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99 As of September 14, 1769 (Lalanne 1882, p. 147 note 1).

100 As of October 5, 1777 (Holcroft 1789b, Vol. 12, p. 107). The topic he proposed to discuss was the question of «whether it be useful to deceive the people», of which he said:

> We have never dared to propose this great question to the French Academy, because the dissertations, sent for the prize, must, to the misfortune of reason, undergo censorship by two doctors of the Sorbonne; and because it would be impossible, with people like these, to write any thing rational. But your majesty has neither prejudices nor doctors of the Sorbonne. (as of September 22, 1777 (Holcroft 1789b, Vol. 12, p. 104))

101 R. Wolf in Sheynin (2010, p. 168). These and other friends were the people closest to Lambert; he did not marry or have children.

102 Formey in Sheynin (2010, p. 148).

tuberculosis, having never wanted, except in the last 15 days, to take any remedy or consult any doctor. He had received from nature an admirable character and temperament; always satisfied with himself, he never showed the slightest envy or jealousy. He had a very naïve way of thinking and acting, which often turned against him people who did not know him particularly; but, when it had been possible to know him thoroughly, one could not help but conceive for him all the esteem and friendship he deserved; this is what happened to me. If I envy his life, I also envy his death, which has been one of the sweetest, and of which he did not even suspect.[103]

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[103] As of October 3, 1777 (Lalanne 1882, pp. 333–334).

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Part II
Elías Fuentes Guillén

Chapter 2
Lambert, the Circle-Squarers and π: Introduction to Lambert's *Vorläufige Kenntnisse*

> *Twenty-two years I have been trying to find the fixed point.* [...] *The same thing happens to me with the quadrature of the circle, which I have been so close to finding, that I do not know, nor can I conceive, how I do not already have it in my pocket.*
>
> —Miguel de Cervantes, *The Dialogue of the Dogs.*

> [*T*]*he pursuit of mathematics is a divine madness of the human spirit.*
>
> —Alfred North Whitehead, *Science and the Modern World.*

The treatise «Preliminary Knowledge for Those Seeking the Quadrature and Rectification of the Circle» («Vorläufige Kenntnisse für die, so die Quadratur und Rectification des Circuls suchen») was written in 1766 and published in 1770 as part of the second of the three volumes entitled *Contributions to the Use of Mathematics and Its Application* (*Beyträge zum Gebrauche der Mathematik und deren Anwendung*).[1] According to what Lambert wrote in the preface to this volume, his treatise was intended for «searchers for the quadrature of the circle»,[2] i.e., those who sought to square the circle «with straight edge and compasses» or, ultimately, by means of algebraic curves.[3]

[1] The original version can be consulted at http://www.kuttaka.org/~JHL/L1770a.html.

[2] Lambert (1770, p. [II]).

[3] Klein (1897, pp. 55, 78).

E. Dorrego López and E. Fuentes Guillén, *Irrationality, Transcendence and the Circle-Squaring Problem*, Logic, Epistemology, and the Unity of Science 58,
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Now, unlike the memoir that Lambert presented the following year and in which he included the —as far as we know— first rigorous demonstration of the irrationality of π, his treatise of 1766 is a «semipopular and witty exposition»[4] about the impossibility of obtaining a finite decimal representation of π. Eventually, with the proof of the transcendence of π, the impossibility of constructing —using only compass and ruler— a square the area x^2 of which is equal to the area $r^2\pi$ of a circle, with $x = r\sqrt{\pi}$, was to be established.[5] But a century before this, in the present work, Lambert shows the impossibility of expressing the ratio of the diameter to the circumference «by means of a rational fraction», which was a key aim of those who were known at the time as «circle-squarers». Therefore, while at the time the adjective «preliminary» was used to describe a text that provided a very first approach on a topic, in this case the expression «preliminary knowledge» (for the circle-squarers) may already have been indicative of the ironic tone of the text.

In the late 17th and early 18th centuries there was a burgeoning of attempts to solve the classical problem of the commensurability of the ratio of the circumference to the diameter of the circle. This was largely due to the spreading of the notion that certain states and scientific institutions were awarding prizes for the resolution of this geometrical problem because of its presumed relevance to the resolution of the problem of determining the «fixed point», or the longitude at sea. As Augustus De Morgan pointed out, such was the number of attempts at the time that from the mid-18th century onwards some institutions, including the French *Académie Royale des Sciences* and the *Royal Society of London*, decided not to examine any further work on the subject.[6]

At the time, the importance of determining longitude at sea resided, first and foremost, in its economic and political consequences. In the face of overseas expansion and the ensuing development of maritime trade, some states, scientific institutions, companies and individuals did in fact start to offer prizes for the resolution of this problem from the second half of the 16th century on.[7] This led to the development of new methods, techniques and instruments, and eventually contributed to the more accurate calculation of longitude at sea from lunar distances (with better observational and navigational instruments, as well as improved tables of the distance between the moon and other celestial bodies) and the construction of more precise marine timepieces.[8] But this also led many people to associate the problem of longitude at sea with that of the quadrature of the circle, since —at best— it was assumed that solving the latter would contribute to solving the former, for example by

[4] Ferreirós (2015, p. 218).

[5] Arndt et al. (2001, p. 7).

[6] cf. De Morgan (1872/2015, p. 97). For a detailed account of this issue, cf. Jacob (2005). I am grateful to one of the anonymous reviewers for bringing the latter work to my attention. Cf. also Jacob (2006).

[7] cf. Green (1766, pp. 18–20, 33–37, 66–67), Betts (2018, pp. 5–6, 13–14).

[8] cf. Dunn et al. (2014).

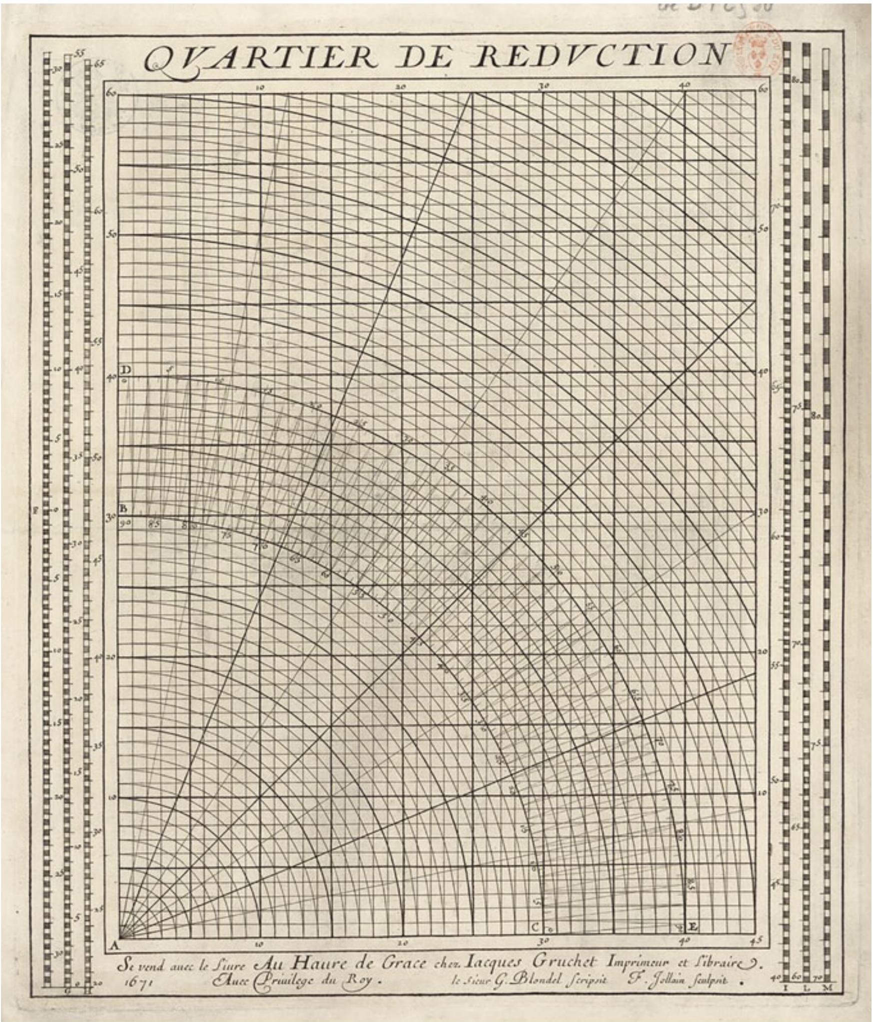

Fig. 2.1 *Quartier de réduction* (1671), engraving by François Jollain (available at https://data.bnf.fr/atelier/14952804/francois_jollain/)

improving the instruments known as «quadrants», such as the quadrant of reduction or sinical quadrant (see Fig. 2.1), which were commonly used to tackle the problem of longitude at sea.[9]

[9] cf. Boistel (2016, pp. 66–67, 446). For a guide on how to use the quadrant, see http://www.meridienne.org/atelier/instruments/quartier-reduction/utilisation/ (accessed on 26 March 2024).

The confusion which ensued here was at least partly due to the fact that some encyclopaedias and dictionaries spread the idea that some of those states which were offering prizes for solutions to the problem of longitude at sea were in fact offering these prizes just for solving the problem of the quadrature of the circle.[10] This was not the case, even though it is true that some prizes linked to this latter problem were indeed offered. Thus, while there were people who offered prizes for the refutation of the alleged solutions to this problem,[11] Jean-Baptiste Rouillé de Meslay allocated a sum of money for research about it in his will, although the French *Académie Royale des Sciences* ended up re-allocating it to research on navigation.[12]

The works mentioned by Lambert in §. 3 precisely form part of the corpus produced by circle-squarers during the 18th century. Moreover, these works not only coincide in their purpose but also in the specific ratio which they proposed for the circumference of a circle to its diameter, namely $3844 : 1225$, obtained from $\frac{4 \cdot 35^2}{31^2}$. It follows, then, as Lambert notes (§§. 2–3), that Joseph Ignatius Carl von Leistner (in 1737 and 1740), Johann Christoph Merkel (in 1751) and, following the latter, Johann Christoph Bischof (in 1765) all advocated a much less precise approximation of π than others well known at the time, starting with that by Archimedes,[13] according to whom

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7},$$

which dates from the third century BCE[14] and is accurate to two decimal places, or that by Ludolph van Ceulen[15] for the first 32 decimal places of the lower and upper bounds of π:

$$\text{Lower bound} : 3\frac{14159265358979323846264338327950}{100000000000000000000000000000000}$$

$$\text{Upper bound} : 3\frac{14159265358979323846264338327951}{100000000000000000000000000000000}.$$

In order to discourage the use of such a popular ratio, Lambert explains a «general rule» which might be used to obtain more accurate ratios. First, he says, given $a = 1$ as the diameter of the circle and b as the side of a square with the same area as the former, i.e., $\pi \cdot r^2 = \frac{\pi \cdot a^2}{4} = b^2$, one gets $a^2 : 4b^2 = 1 : \pi$ and therefore $a : b = 2 : \sqrt{\pi}$ or $= \frac{200000000000}{1.77245385075}$. Secondly, he calculates the continued fraction associated

[10] cf. De Messanges (1686, pp. 14–15), Chambers (1728, p. 221), Society of Gentlemen (1754, p. 593).

[11] cf. De Causans (1754), Hutton (1815, pp. 273–274).

[12] cf. Montucla (1802, p. 384), Boistel (2016, pp. 46–48).

[13] cf. Heath (1897, p. 93).

[14] cf. Knorr (1993, pp. 153–155).

[15] cf. Van Ceulen (1615, p. 163). Because of the accuracy of this approximation, π was for a long time known as the «Ludolphian number» (*Ludolphsche Zahl*), especially in German-speaking territories, and in fact there are still people today who know it by this designation, cf. Arndt et al. (2001, p. 183).

with the decimal development of this quotient up to the seventh row and from this he obtains the sequence of rationals $\frac{b}{a} = \frac{7}{8}, \frac{8}{9}, \frac{31}{35}, \frac{39}{44}, \frac{109}{123}, \frac{148}{167}, \frac{3845}{4342}$, etc., which are «more precise according to their order» (§. 4).

As Lambert points out, such fractions express the side of a square with the same area as a circle, the diameter of which is assumed to be $= 1$, so that, inverted, they express the diameter of a circle the area of which is $= 1$ (§. 5), thus providing a calculation the margin of error of which is insignificant for certain practical matters: for such a circle, the diameter of which is 1.128379..., the fraction $\frac{35}{31}$ approximates with a difference of 0.00065..., the fraction $\frac{44}{39}$ approximates with a difference of 0.00017..., and so on. Finally, he addresses the case of cube numbers, used for the comparison of the diameter of a sphere with the side of a cube with the same volume as the former, obtaining the sequence of rationals from the continued fraction associated to $\sqrt[3]{\frac{\pi}{6}}$ (§. 6), and explains the procedure which he followed in order to obtain this cube root and to verify that it is correct up to the 18th decimal place (§§. 8–9).

Lambert then goes on to discuss the problem of «whether the ratio of the diameter to the circumference can be expressed by means of a rational fraction» (§. 10) and presents a sequence of 27 ratios obtained following a method used in his treatise «Transformation of Fractions» («Verwandlung der Brüche», also included in the second volume of his *Contributions*).[16] As he explains, each of these ratios «is more exact than the preceding one» (§. 10), which means that any rational proposed as the exact value for the ratio of the diameter of a circle to its circumference should therefore be greater than the last ratio provided by him, namely $\frac{1019514486099146}{3245215400329 45}$, which, without this affecting his argument, is incorrect along with his 26th ratio. Lambert himself acknowledges that his ratios correspond to the Ludolphian «numbers» only up to the 25th decimal place and gives the continued fraction from which he obtained his 27 ratios, noting that in his other treatise he gives a continued fraction «which continues to infinity, according to a certain law, and completely removes the hope of determining the ratio of the diameter to the circumference by means of whole numbers» (§. 10).

Precisely, after showing that neither e nor e^x, with x rational, «can be expressed exactly by a rational» (§. 11), Lambert presents in §. 12 the continued fraction for the function $\tan v$:

$$\tan v = \cfrac{1}{1 : v - \cfrac{1}{3 : v - \cfrac{1}{5 : v - \cfrac{1}{7 : v - \cfrac{1}{9 : v - \text{ etc.}}}}}}$$

from which, given an integer n and $v = \frac{1}{n}$, he obtains:

[16] The original version can be consulted at http://www.kuttaka.org/~JHL/L1770a_3.pdf (accessed on 26 March 2024).

$$\tan v = \cfrac{1}{n - \cfrac{1}{3n - \cfrac{1}{5n - \cfrac{1}{7n - \cfrac{1}{9n - \cfrac{1}{11n - \text{ etc.}}}}}}}$$

As Lambert notes, since this fraction continues to infinity, the tangent of a rational circular arc «will necessarily be irrational» (§. 12), a conclusion which he explains in detail in the subsequent sections (§§. 12–14). In addition, he notes that from this it follows that, «conversely, the arc of every rational tangent is irrational» (§. 15), as well as that, in the case of the arc of 45°, it has «no rational ratio to the radius of the circle» (§. 16). Lambert did not explain further the consequences of his results, but what was entailed by these latter was that $\frac{\pi}{4}$ was irrational and, ultimately, that π was irrational as well. In fact, in his treatise «Transformation of Fractions» he had already given the continued fraction for the arc of 45°, which he obtained from the series $\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} - \text{etc.}$[17]

Finally, Lambert goes on to state the impossibility of the radius, the arc and the tangent being all commensurable at the same time, so that if the latter «have a rational ratio to each other, then both are *incommensurable* with the radius» (§. 17), and presents to his readers the following «*phenomenon*» in §. 18:

> if one divides 1 by 0, 7853981634..., as a fourth part of the **Ludolphian** numbers, it occurs 1 time and subtracts 0, 2146018366... If one further divides by this remainder 0, 7853981634..., which was previously the divisor, then it occurs 3 times and subtracts 0, 1415926536... If one places the number 3 in front of this remainder, one gets 3, 1415926536..., which are precisely the **Ludolphian** numbers.

Lambert does not explain the «cause» of this «phenomenon» and merely warns that nothing can be concluded from it regarding the quadrature of the circle, the point here being rather just that, given $\pi = 3 + x$, in order to find x one sets $\frac{1}{\frac{\pi}{4}} = 1 + \frac{r}{\frac{\pi}{4}}$, from which $r = 1 - \frac{\pi}{4}$, and $\frac{\frac{\pi}{4}}{r} = 3 + \frac{x}{r}$, from which $x = \frac{\pi}{4} - 3r$, so that $x = \frac{\pi}{4} - 3 + \frac{3\pi}{4}$ and, therefore, $x = \pi - 3$.[18]

Over the next hundred years, frequent attempts continued to be made to square the circle and to find the rational value of π. These included the following notorious cases: during the 1770s Alexandre-Henry-Guillaume le Roberger de Vausenville made several attempts to have his quadrature of the circle either recognised or refuted and even went so far as to sue the French *Académie Royale des Sciences* and demand that the aforementioned prize instituted in honour of de Meslay be awarded to him;[19] in 1836 Joseph LaComme, a peasant artisan who, in trying to determine «the amount

[17] Lambert (1765–1766/1770, p. 82), cf. Bauer (2005).

[18] Unger (1829, pp. 326–327).

[19] Schepler (1950a, p. 225), De Vausenville (1778).

of stone required to pave the circular bottom of a well», found out about the problem of the ratio of the diameter of a circle and its circumference, decided to learn mathematics on his own and to focus fully on solving this problem, coming to the conclusion that the exact ratio was 25 : 8, a result for which he eventually achieved some recognition;[20] and from 1859 on James Smith not only published several works in which he also asserted that the «true value» of such a ratio was 3.125, but in addition to this corresponded with several mathematicians, such as De Morgan and William Whewell, who tried, unsuccessfully, to make him see his error, as he himself revealed by publishing some of these letters.[21]

Moreover, even after Carl Louis Ferdinand von Lindemann's 1882 proof of the transcendence of π (i.e., that it cannot be the root of a polynomial with rational coefficients and therefore it is not possible to square the circle «with straight edge and compasses»), there continued to be claims raised about the rationality of π, such as $3 + \frac{13}{81}$ in 1934 and 3.1428 in 1983,[22] and about the possibility of the quadrature of the circle. However, whereas in the first case it has been established that this is a perennial quest, one equated in Cervantes' *The Dialogue of the Dogs* with the punishments of Tantalus and Sisyphus, in the second case a number of results obtained during the last three decades illustrate how the reformulation of a problem within a different framework can lead to new ideas: Tarski's circle-squaring problem, for example, asks if a disc in the plane $\mathbb{R}^2$ is equidecomposable with a square of the same area (i.e., if the former can be decomposed into finitely many pieces which can be reassembled to obtain a partition of the latter), and Laczkovich (1990), Grabowski et al. (2016/2020), Marks et al. (2017) and Máthé et al. (2022) have all proven in different ways that it is possible to do so. For better or for worse, then, a certain venturing into endeavours deemed impossible is not only inherent to mathematical practice but actually enriches this latter. Lambert's own work is an example of this.

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[20] Schepler (1950a, p. 226), cf. Blanc (1997, pp. 12–13).

[21] Schepler (1950a, pp. 227–228), De Morgan (1872/2015, pp. 44–51), cf. Smith (1869), cf. Smith (1870).

[22] Schepler (1950b, p. 283), Arndt et al. (2001, p. 8).

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Chapter 3
An Annotated Translation of Lambert's *Vorläufige Kenntnisse* (1766/1770)

As far as I know, it has not yet been elucidated whether the ratio of the diameter to the circumference can be expressed by means of a rational fraction. [...] *Since the matter therefore remains to be elucidated, there may still be people who waste their time searching for such rational fractions or who bring them up as a consequence of erroneous conclusions.*

—J. H. Lambert, *Vorläufige Kenntnisse.*

Preliminary Knowledge for Those Seeking the Quadrature and Rectification of the Circle.

§. 1.

I have some reason to doubt whether the present treatise will be read or even understood by those who should take the most interest in it, I mean by those who spend time and effort in seeking to square the circle. There will certainly always be plenty of such people, and if those who occupy themselves with this matter in the following times were to be judged on the basis of those who have occupied themselves with it hitherto, they will mostly be those who barely understand geometry and are incapable of assessing their strengths. However, what most of these people lack in knowledge, understanding, and correct and coherent conclusions, their lust for fame and money replaces with *sophismata*, which are often neither very subtle nor well hidden. There have also been cases in which such people have firmly believed that their supposed proofs were being denied acclaim merely out of envy and resentment. There is also a legend circulating amongst them that in England and Holland equally big prizes and rewards were set for the quadrature of the circle as for the finding of the geographical longitude at sea. I certainly do not intend to vouch for whether or not it was believed at the beginning of the last century, or even before, that the finding of the longitude at sea had such a connection with the quadrature of the circle that whoever found

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the latter would also have found the former. What is certain is that, at that time, people sought for and believed in relations between truths that were even less likely to fit together. However, should a prize have indeed been set for the quadrature of the circle because of the longitude at sea, I believe that the Parliament of England would do well to announce in all the newspapers that no one should reckon with a prize being given for the quadrature of the circle, especially since the prizes for the longitude at sea have already been allotted. In fact, one should indeed not count on it, because nowadays it is known far too well how independent the longitude at sea is from the quadrature of the circle.

§. 2.

The finding of things which have long been sought in vain is either impossible per se or its occurrence is reserved for some future happy coincidence. An example may illustrate this. There is no doubt but that the ancient Phoenicians, and after them the Greek and Roman mariners, desired a means which would show them the proper route of the ship on a cloudy day just as the stars showed it on a clear one. How could it have occurred to them that the place to look for such a means was in lodestones? It is indisputable that this discovery depended on an absolutely unforeseeable confluence of circumstances which could not have been arranged without foreknowledge and which therefore had to come to pass on its own. Likewise, it is to be assumed that if the quadrature of the circle were possible at all, the means of performing it would perhaps occur to a practical geometer[1] from whose mind there lay nothing farther than the discovery of such a thing. However, it is just as possible to accidentally arrive at erroneous quadratures. The numbers 1225 and 961 provide a good example of this. They have a twofold characteristic: on the one hand, they are the square numbers of 35 and 31; on the other hand, they stand almost in the same relation to one another as does the square of the diameter to the content of the circle. This in turn means that the diameter of the circle is in relation to the side of a square spatially equal to the circle almost as 35 is in relation to 31. Thus, if one quadruples 961, one obtains 3844, which is also a square number, and the diameter will relate to the circumference almost as 1225 does to 3844. But this **almost** must not be taken very strictly, because if one divides 3844 by 1225, one gets 3, 138... And it is easy to see that this ratio deviates from 3, 1415926... already in the 3rd decimal place and is therefore not nearly as exact as the Archimedean ratio 22 : 7,[2] which gives the sequence 3, 1428571..., a sequence which is only 0, 0012645 larger and, for this reason, almost three times more exact.

§. 3.

Nevertheless, the numbers 1225 and 961, or 1225 and 3844, retain a certain value, as they are square numbers. In this century, as far as I know, three authors have arrived

[1] The term used by Lambert is «Meßkünstler», which was used at the time to refer to «one who understands and practices the art of measurement» (Campe 1809, p. 275), that is, a «geometer» or, more precisely in this case, a «practical geometer» or *mensor* (Grimm and Grimm 1885, col. 2137).

[2] Lambert refers to the 3rd proposition of Archimedes' treatise entitled *Measurement of a Circle: «The ratio of the circumference of any circle to its diameter is less than* $3\frac{1}{7}$ *but greater than* $3\frac{10}{71}$» (Heath 1897, p. 93).

at them. This situation seems to me very strange. Because, since there are several more such square numbers, one would rather think that each of these three inventors would have arrived at different numbers. The first was Mr von **Leistner**, Captain of the Imperial Cavalry. He found the numbers 1225 and 3844,[3] which were adjudged incorrect by the Imperial Court Commission, against which, however, he protested in *Nodus Gordius* &c., a writing published *anno* 1740.[4] The other was Mr **Merkel**, a preacher in Ravensburg,[5] Schwaben, whose writing did not come to light until *anno* 1751.[6] He claims, however, that he found his numbers 1225 and 961 by chance, long before Mr von **Leistner**, but was only persuaded to put them to the test by the *Nodus Gordius*, although what particularly prompted him to bring them to public notice by printing them was an article in Utrecht's newspaper, in which a quadrature was announced and the prize supposedly set on it was demanded. This news led him to lend more speed to the pen in his hand, because the previous winter he had run through his own calculations for a Frenchman who had in fact later travelled to the Netherlands, without paying further attention to the paper, and so he had strong reasons to suspect that this *geometra* may already have «ploughed the land with his calf»,[7] etc. What happened afterwards is unknown to me. However, Merkel's writing was reissued *anno* 1765 by Prof. **Bischoff** of Alten-Stettin, with annotations and several further proofs,[8] and the numbers 1225 and 961 were declared correct. Shortly afterwards, at the beginning of 1766, these numbers appeared again in the

[3] These numbers can be found, for example, in Joseph Ignatius Carl von Leistner's *Unwiderruf-flicher, Wohlgegründter und Ohnendlicher Beweiß der Wahren Quadratur des Circuls, oder des Durchmessers zu seinem Umcreyß, wie 1225 zu 3844 oder 3844 zu 1225* (Von Leistner 1737).

[4] The full title of von Leistner's work is *Der durch Kunst und Wissenschaft eröfnete* Nodvs Gordivs. *Das ist: Kurtzer und unpartheyischer Bericht, von der ohnlängst herausgekommenen, nunmehro zwar vor wahr gehaltenen, jedoch in den letzten Zügen gelegenen, aber jetzo wieder aufs neue erstandenen* Quadratura Circuli (Von Leistner 1740).

[5] In the original there is a mistake here, as it says «Rakensburg» (Lambert 1766/1770, p. 144).

[6] The title of Johann Christoph Merkel's work is *Die Wirklichkeit der* Quadratur *des Cirkuls, in der* Proportion *des* Quadrati Diametri *zu dem Innhalt des Cirkuls, wie 1225 zu 961* (Merkel 1751). As noted by Ferdinand Rudio, Gotthold Ephraim Lessing refers to Merkel in his epigram «Auf den Herrn M** den Erfinder der Quadratur des Zirkels», published in 1751 (Lessing 1886, p. 38), (Rudio 1892, p. 137).

[7] The expression «mit eines Andern Kalbe pflügen» was customary at the time, cf. Campe (1809, p. 638). Here the abbreviation for «et cetera» was typeset in Fraktur typeface (like the rest of the text) using, on the one hand, the glyph for a rounded r (similar to a capital R but without the stem, i.e., the main vertical stroke) instead of the glyph for the Tironian *et* (similar to a 7), and, on the other hand, the glyph for a «c». Whereas in this translation, as in the rest of the book, we have tried to adhere to the notation used in the originals, here I have opted to replace this abbreviation by «etc.».

[8] Lambert refers to *Johann Christoph Merckels Evangelischen Predigers zu Ravensburg in Schwaben Beweis von der Würcklichkeit der Quadratur des Circkels in der* Proportion *des* Quadrati diametri *wie 1225 zu 961. Untersuchet und mit Anmerckungen versehen* (Bischof 1765), published by Johann Christoph Bischof, professor of mathematics and physics at the Royal Gymnasium in Alten-Stettin.

newspapers, with the solemn announcement that it was no longer necessary to search for the quadrature of the circle, since it had already been found, and indeed it had already been found for the third time, etc. It would not be a bad thing if many of the people who set to work on this matter in the future believed this very strongly, as they would thereby be spared the waste of effort, time and strength, which can be regarded as having been used in vain, since most of them are hardly capable of contriving and solving a simple geometrical problem. But there is little doubt but that **Merkel's** and **Leistner's** numbers will be rehashed at some point in the future. The main proof of their inaccuracy is that 3844 divided by 1225 should give the **Ludolphian** numbers.[9] Professor **Bischoff** also considers these **Ludolphian** numbers, and even **Sherwin's** numbers,[10] which are more than twice as far-reaching, although he does not consider them as touchstones, but rather says that while they are very close, they do not give the content of the circle with complete accuracy, hence other proofs are to be considered. Mr **Bischoff** carries out 8 such proofs and thus makes the matter seem plausible. It is indisputable that if one were to divide 3844 by 1225, thus giving exactly the **Ludolphian** numbers, which reach up to 32 decimal places, one could on the one hand be very satisfied with this, but on the other hand one would have to see whether this division would also give **Sherwin's** numbers, which reach up to 72 decimal places, and then **Machin's** numbers, which reach up to 100 decimal places,[11] or finally de **Lagny's** numbers, which reach up to 127 decimal places.[12] One could, then, be all the more satisfied with the proportion 3844 : 1225. However, as soon as the above-mentioned division is carried out, the quotient 3, 138... already starts to deviate from the Ludolphian numbers in the third decimal place. And, furthermore, the 8 proofs are such that any two square numbers can bear them. However, I will not dwell on showing this here, but rather indicate how, according to a general rule, such square numbers, which indicate the ratio of the square of the diameter to the content of the circle, can be established more accurately the larger they are. This may also serve, among other things, to ensure that in the future it will no longer be necessary to arrive at such square numbers in a merely random manner and then to present them as completely correct quadratures of the circle.

[9] What Lambert calls the "Ludolphian numbers" owe their name to Ludolph van Ceulen (1540–1610), who calculated the first 32 decimal places of the lower and upper bound of π and reportedly even have calculated the first 35 decimal places, cf. Van Ceulen (1615, p. 163), Arndt et al. (2001, p. 183).

[10] Lambert refers here to the 72 decimal places of π calculated in the *Mathematical Tables* edited by Henry Sherwin, cf. Sherwin (1706, p. 57).

[11] William Jones included the calculation of π by John Machin (1686–1751) in Jones (1706, p. 243). The procedure used by Machin to find the quadrature of the circle is described in Maseres (1758, pp. 289–293).

[12] In the original there is a mistake here, as it says «Lamysche» (Lambert 1766/1770, p. 146), i.e., «of [de] Lamy», to refer to Thomas Fantet de Lagny (1660–1734), who in his *Mémoire sur la Quadrature du Cercle, & sur la mesure de tout Arc, tout Secteur, & tout Segment donné* (De Lagny 1719/1721) calculated the value of π referred to by Lambert. Here, as in Rudio (1892, p. 138), this mistake has been corrected.

§. 4.

Consider two square numbers aa, bb, such that if a is the diameter of the circle and therefore aa is its square, then bb represents the content of a square spatially equal to the circle, and therefore, b represents the side of the circle. In this way, aa stands in the same relation to $4bb$ like the diameter with respect to the circumference, or like 1 with respect to 3, 141592, 653589, 793238, 462643, 383279, 502884, 197169, 399375, 105820, 974944, 592307, 816406, 286208, 998628, 034825, 342117, 067982, 1480 86, 513272, 306647, 093844, $6 + \cdots = 1 : \pi$.[13] According to this, $aa : 4bb = 1 : \pi$ and from this follows

$$a : b = 2 : \sqrt{\pi}.$$

But $\sqrt{\pi} = 1,77245385075....$ And from this one gets

$$a : b = \frac{2,00000000000}{1,77245385075} = 1 + \cfrac{1}{7 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{26 + \text{etc.}}}}}}}}$$

This gives, in order,

$$\begin{array}{llcll} b : a & = 7 : 8 + \cdots & \text{and} & bb : aa & = 49 : 64 + \cdots \\ & = 8 : 9 - \cdots & & & = 64 : 81 - \cdots \\ & = 31 : 35 + \cdots & & & = 961 : 1225 + \cdots \\ & = 39 : 44 - \cdots & & & = 1521 : 1936 - \cdots \\ & = 109 : 123 + \cdots & & & = 11881 : 15129 + \cdots \\ & = 148 : 167 - \cdots & & & = 21904 : 27889 - \cdots \\ & = 3845 : 4342 + \cdots \ \text{etc.} & & & = 14807104 : 18852964 + \cdots \ \text{etc.} \end{array}$$

These fractions are therefore more precise according to their order. Moreover, it can be seen from this that Messrs von **Leistner**, **Merkel**, **Bischoff** etc., arrived only accidentally at their numbers 961 and 1225. After all, the calculation with 49 : 64 or 64 : 81 would have been much easier and shorter, whereas with 1521 : 1936 or 11881 : 15129 etc. it would have been more extensive but at the same time more exact.

§. 5.

It is nevertheless more advisable in general to use only the first of these ratios, namely $b : a$. For $bb : aa$ one has other fractions which, without being square numbers, are

[13] In De Lagny (1719/1721) there is a mistake in the 113th decimal digit, which should be 8 instead of 7, although this error could be attributable to transcription (Arndt et al. 2001, p. 193).

much smaller and much more exact, so that, by proceeding in this way, one arrives at

$$\begin{aligned} bb : aa = \pi : 4 &= 11 : 14 \\ &= 172 : 219 \\ &= 355 : 452 \text{ etc.} \end{aligned}$$

However, the fractions $\frac{7}{8}$, $\frac{8}{9}$, $\frac{31}{35}$, $\frac{39}{44}$, $\frac{109}{123}$, $\frac{148}{167}$, $\frac{3848}{4342}$ etc. express the side of a square which is as large as the area of a circle, the diameter of which is assumed to be $= 1$. And conversely, the same fractions placed upside down, or $\frac{8}{7}$, $\frac{9}{8}$, $\frac{35}{31}$, $\frac{44}{39}$, $\frac{123}{109}$, $\frac{167}{148}$, $\frac{4342}{3848}$ etc., represent the diameter of a circle, the content of which is $= 1$. In this sense, they can be used in the measurement of cylinders and in the manufacture of gauging rods.[14] The fraction $\frac{167}{148}$ is particularly useful for this purpose, since it is the most accurate of the smaller ones and only begins to deviate from the correct decimals from the seventh place onwards. Because, if one calculates it according to the **Ludolphian** numbers, the diameter of the circle, the content of which is $= 1$, is found $= 1,1283790\ldots$ However, $\frac{167}{148}$ is $= \underline{1,1283784...}$, so the difference is $= 0,0000006\ldots$ It rarely happens that in practical cases one needs to know this diameter more precisely.

§. 6.

Since it is also possible, when comparing the diameter of a sphere with the side of a spatially equal cube, to succumb to the temptation of accepting such cubic numbers, on the basis of which one might dream of the quadrature of the circle or the cubature of the sphere, it will therefore surely prove to be useful in preventing such future incidents to determine these cubic numbers in advance by the same method, especially since they can be used advantageously when calculating the spatial content of the spheres and when manufacturing the calibre-rods.[15] Thus, let the diameter of the sphere be $= a$, the side of the spatially equal cube $= b$, and the **Ludolphian** numbers $3,1415926\ldots = \pi$, so according to the well-known **Archimedean** rule,

$$b^3 : a^3 = \pi : 6$$

[14] Lambert uses the expression «cylindrisch[e] Visirstäbe» (Lambert 1766/1770, p. 149) to refer to the cylindrical rods used to measure the volume of casks (Grimm and Grimm 1926, col. 376), cf. Büsch (1776, pp. 141–144). I am grateful to one of the anonymous reviewers for her/his helpful remarks on this issue. Interestingly, in the second edition of his book, Johann Georg Büsch (1728–1800) notes that «reliable rules» for the measurement of casks were given by Lambert in «Die Visirkunst, sowohl ganz als nicht ganz angefüllter liegender Fässer, auf ihre einfachsten Gründe und Regeln gebracht» (Büsch 1776, p. 143). This work was included in the first volume of Lambert's *Contributions* (Lambert 1765, pp. 314–368), the third volume of which includes some «additions» on the subject (Lambert 1772, pp. 12–34).

[15] The «calibre-rods» («Caliberstäbe») were used to measure bullets: «Calibre-rod, or artillery measuring-rod, is a rod on which the diameters of iron, stone, or leaden balls of different weights are marked, so that the calibre or diameter of a gun or cannon being known, one can discover the weight of the iron, stone, or leaden ball which it will carry. The weight of the ball also being given, it serves to determine the calibre of the piece» (Beckmann 1817, p. 461).

and therefore

$$b : a = \sqrt[3]{\frac{\pi}{6}}.$$

Hence, it follows that

$$\begin{aligned} \pi &= 3,141592,653589,793238,462... \\ \frac{1}{6}\pi &= 0,523598,775598,298873,077... \end{aligned}$$

And from this one gets the cubit root

$$b : a = 0,805995,977008,234820...,$$

which, developed into a continued fraction, gives

$$b : a = \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{6 + \cfrac{1}{2 + \cfrac{1}{8 + \cfrac{1}{6 + \cfrac{1}{6 + \text{etc.}}}}}}}}$$

From which one gets

$$\begin{aligned} b : a &= 4 : 5 + \\ &= 25 : 31 - \\ &= 54 : 67 + \\ &= 457 : 567 - \\ &= 2796 : 3469 + \\ &= 17233 : 21381 - \quad \text{etc.} \end{aligned}$$

Accordingly, if the diameter of a sphere is $= 1$, the side of a spatially equal cube is expressed with greater precision by each of the fractions $\frac{4}{5}$, $\frac{25}{31}$, $\frac{54}{67}$, $\frac{457}{567}$, $\frac{2796}{3469}$, $\frac{17233}{21381}$ etc. the larger they are. If these fractions are raised to the cube, they give the content of the sphere. But if one sets the physical content of the sphere $= 1$, then these same fractions placed upside down, $\frac{5}{4}$, $\frac{25}{31}$, $\frac{67}{54}$, $\frac{567}{457}$, $\frac{3469}{2796}$, $\frac{21381}{17233}$ etc.,[16] represent the diameter of the sphere. In general, one can be satisfied with the fraction $\frac{567}{457}$, since,

[16] At this point Lambert made a mistake in the second fraction, which should be $\frac{31}{25}$ (Lambert 1766/1770, p. 151), (Rudio 1892, p. 142).

if recalculated, it gives the diameter of the sphere with the same precision as if one had arrived at it by means of the logarithmic tables.

§. 7.

In this calculation, I have supplied the cubic root of $\frac{1}{6}\pi$ up to the 18th decimal place. Since it would be a tedious and extremely time-consuming task to search for it, as hitherto, according to the common rules, it will surely be useful if I add how I found this root by means of a single rule *de tri* and, at the same time, how I made sure that it is correct up to the 18th decimal place.

§. 8.

According to **Newton's** binomial formula, it follows that, in general,

$$x = (a+b)^n = a^n + na^{n-1}b + n \cdot \frac{n-1}{2} \cdot a^{n-2}b^2 + \text{ \&c.}$$

Now multiply this series by $1 + zb : a$, and in the product

$$\begin{aligned} x\left(1+\frac{zb}{a}\right) = a^n &+ na^{n-1}b + n \cdot \frac{n-1}{2}a^{n-2}b^2 \\ &+ n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3}a^{n-3}b^3 + \text{etc.} \\ &+ za^{n-1}b + n \cdot z \cdot a^{n-2}b^2 + n \cdot \frac{n-1}{2} \cdot za^{n-3}b^3 + \text{etc.,} \end{aligned}$$

in order to determine z, set the third term

$$n \cdot \frac{n-1}{2} \cdot a^{n-2}b^2 + n \cdot z \cdot a^{n-2}b^2 = 0$$

so that

$$z = -\frac{n-1}{2}.$$

If one now places this value of z in the product, one obtains[17]

$$x\left(1 - \frac{n-1}{2} \cdot \frac{b}{a}\right) = a^n + \frac{n+1}{2}a^{n-1}b + * - n\frac{n-1}{2} \cdot \frac{n+1}{6}a^{n-3}b^3 - \text{ \&c.}$$

and from this

$$x = (a+b)^n = \left(\frac{2a+(n+1)b}{2a-(n-1)b}\right)a^n + * - \frac{n \cdot (n-1) \cdot (n+1)a^{n-2}b^3}{6\,(2a-(n-1) \cdot b)} - \text{ \&c.}$$

[17] In Rudio (1892, p. 143), «$+ * -$» was replaced by «$-$» in this and the next formula.

Of this series, the first term is used to determine the root, while the second term is used to find out how far one can go with the first one.

§. 9.[18]

Now, for the cubic root one has $n = \frac{1}{3}$. If one sets this value, one gets, after the relevant reductions, the formula[19]

$$x = \sqrt[3]{a+b} = \frac{3a+2b}{3a+1b} \cdot \sqrt[3]{a} + * + \frac{2b^3 \cdot \sqrt[3]{a}}{81a^3 + 27aab} + \ \&c.$$

I have applied this as follows to extract the cubic root of

$$a + b = \frac{1}{6}\pi = 0, 523598, 775598, 298873, 077...$$

First, using the logarithms, I found the first six decimal places of this root. These are

$$0, 805995 = \sqrt[3]{a}.$$

And, since

$$805995 = 806000 - 5,$$

the cube should then be easy to find from this. I therefore set it as

$$0, 523596871520449875 = a$$

and thereby obtained

$$b = 0, 0000019040778489980771 07...$$

Now, since if one only retains the first term of the series

$$x = \sqrt[3]{(a+b)} = \frac{3a+2b}{3a+b} \cdot \sqrt[3]{a}$$

this gives by the rule *de tri*

$$(3a+b) : (3a+2b) = \sqrt[3]{a} : x$$

or

[18] In the original, this section appears as the 8th, instead of the 9th, as a consequence of which the numbering of the subsequent sections is incorrect by one number. Here, as in Rudio (1892, pp. 143ff.), this mistake has been corrected.

[19] In Rudio (1892, p. 144), «$+ * +$» was replaced by «$+$» in the next formula.

$$\left(a+\frac{1}{3}b\right):\left(a+\frac{2}{3}b\right)=\sqrt[3]{a}:x,$$

I only had to set the values of a and b in order to obtain the value of

$$x=\sqrt[3]{(a+b)}=\sqrt[3]{\frac{\pi}{6}}=0,805995977008234820...$$

However, I found that this value is correct up to the 18th decimal place by means of the second term of the series

$$\frac{2b^3\cdot\sqrt[3]{a}}{81a^3+27a^2b}$$

thanks to a simple estimation. Because, given that b is 275000 times smaller than a, I was able to set this term as

$$\frac{2b^3}{81a^3}\cdot\sqrt[3]{a}.$$

Hence[20]

$$\begin{aligned}3\log.b:a&=0,6820508-17\\ \log.\frac{2}{81}&=\underline{1,6074550}\end{aligned}$$

and therefore

$$\log.\frac{2b^3}{81a^3}=0,9145458-19.$$

Now, since in this case the *characteristica* is $=-19$, and a is <1, it is clear that

[20] Rudio introduced the following amendments:

Hence

$$\begin{aligned}3\log\frac{b}{a}&=0,6820631-17\\ \log\frac{2}{81}&=\underline{0,3925450-2}\end{aligned}$$

and therefore

$$\log\frac{2b^3}{81a^3}=0,0746081-18.$$

Furthermore,

$$\frac{1}{3}\log a=\underline{0,9063323-1},$$

therefore

$$\frac{2b^3}{81a^3}\sqrt[3]{a}=0,9809404-19.$$

(Rudio 1892, p. 145).

$$\frac{2b^3}{81a^3} \cdot \sqrt[3]{a}$$

represents a decimal fraction that starts only at the 19th decimal place. Therefore, the decimal sequence found by means of

$$x = \frac{3a + 2b}{3a + b} \cdot \sqrt[3]{a}$$

is accurate up to the eighteenth place.

§. 10.

As far as I know, it has not yet been elucidated whether the ratio of the diameter to the circumference can be expressed by means of a rational fraction. **Sturm**[21] has indeed attempted to answer this question in the negative, but his proof is inadequate, since there are infinite series the sum of which is rational even though all the terms are irrational. Since the matter therefore remains to be elucidated, there may still be people who waste their time searching for such rational fractions or who bring them up as a consequence of erroneous conclusions. It is true that in each case the proof is quickly provided by means of the **Ludolphian** numbers. But even if the given fraction is thereby rejected, the desire to look for others can always persist. This desire may nevertheless become so minimal that one willingly gives up the search for such fractions. For even if the ratio of the diameter to the circumference could be expressed exactly by means of a rational fraction, it can be proven from the de **Lagny's**[22] numbers mentioned above (§. 4), or also from the **Ludolphian** numbers, that it must be a very large fraction. These numbers can be transformed into fractions, which become larger and at the same time more precise according to their order. I have indicated the method and the caution to be exercised in applying

[21] Lambert refers to Johann Christoph Sturm (1635–1703), about whom Rudio wrote the following: «He made himself known by means of excellent books on mathematics and astronomy which are still quite noteworthy today. The study mentioned by Lambert is to be found in the extremely interesting compendium 'Joh. Chr. Sturmii Mathesis enucleata' (Nuremberg, 1689), where on page 181, Prop. XLIII, the following proposition is stated (probably for the first time in this precise form): 'Area circuli est quadrato diametri incommensurabilis'. Sturm was also the first to translate Archimedes' writings into German, which is also of interest to us here. In 1667 he published 'Des unvergleichlichen Archimedis Sandrechnung' and in 1670 'Des unvergleichlichen Archimedis Kunstbücher'. Both translations were published in Nuremberg. The latter contains Archimedes' 'Measurement of a Circle'. See the work of J. G. Doppelmayr (pp. 114–122) cited on page 28» (Rudio 1892, pp. 145–146). Here Rudio refers to Johann Gabriel Doppelmayr's *Historische Nachricht Von den Nürnbergischen Mathematicis und Künstlern* (Doppelmayr 1730).

[22] Here, as in Rudio (1892, p. 146), Lambert's reference to «Lamysche» (Lambert 1766/1770, p. 156) instead of «Lagnysche» has been corrected.

it, and explained it by examples, in §. 17 of the treatise on the **Transformation of Fractions**.[23] Using this method, I found the following rational fractions or ratios for the ratio of the diameter to the circumference:[24]

$$\begin{aligned}
1 &: 3\\
7 &: 22\\
106 &: 333\\
113 &: 355\\
33102 &: 103993\\
33215 &: 104348\\
66317 &: 208341\\
99532 &: 312689\\
265381 &: 833719\\
364913 &: 1146408\\
1360120 &: 4272943\\
1725033 &: 5419351\\
25510582 &: 80143857\\
52746197 &: 165707065\\
78256779 &: 245850922\\
131002976 &: 411557987\\
340262731 &: 1068966896\\
811528438 &: 2549491779\\
1963319607 &: 6167950454\\
4738167652 &: 14885392687\\
6701487259 &: 21053343141\\
567663097408 &: 1783366216531\\
1142027682075 &: 3587785776203\\
1709690779483 &: 5371151992734\\
2851718461558 &: 8958937768937\\
107223273857129 &: 336851849443403\\
324521540032945 &: 1019514486099146 \text{ etc.}
\end{aligned}$$

Of these ratios, each subsequent one is more exact than the preceding one, and among them there is no rational ratio which is more exact than the next larger of those given here. Accordingly, even if the ratio of the diameter to the circumference could be expressed accurately by whole numbers, these numbers must necessarily be greater

[23] The treatise «Verwandlung der Brüche» was also included in the second volume of Lambert's *Contributions to the Use of Mathematics and Its Application* (*Beyträge zum Gebrauche der Mathematik und deren Anwendung*) and was written between 1765 and 1766, as can be deduced from the reference in the present treatise (written in 1766) to it and Lambert's statement in the preface to said volume that all works contained in it were written in or after the year 1765, cf. Lambert (1770, p. [II]).

[24] As pointed out by Johann Schultz, the last two ratios are incorrect, as they should be 44485467702853 : 139755218526789 and 136308121570117 : 428224593349304 (Schultz 1803, pp. 158–159).

than the last numbers given here,

$$324521540032945 : 1019514486099146.$$

These two numbers give the **Ludolphian** numbers up to the 25th decimal place. However, even if they were completely accurate, it is easy to see that it would be tedious and difficult to calculate with them. Incidentally, all these ratios result from the *fractio continua*

$$\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{14+\cfrac{1}{2+a}}}}}}}}}}}}}}$$

where a is[25]

[25] The error in Lambert's last two ratios for the ratio of the diameter of a circle to its circumference is due to the fact that here the 26th quotient is mistakenly assumed to be = 37 instead of = 15. The simple continued fraction expansion of π can be consulted at https://oeis.org/A001203 (accessed on 26 March 2024).

$$= \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{84+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{37+\cfrac{1}{3+\text{ etc.}}}}}}}}}}}}}$$

I have not pursued the calculation of this *fractio continua* beyond the **Ludolphian** numbers. So I will not say whether, if the calculation continues, there is any point at which it must cease and break off. If this were the case, then the ratio of the diameter to the circumference could be expressed by means of whole numbers, though they would need to be tremendously large ones. However, in the treatise on the **Transformation of Fractions** mentioned above (§. 23) I have provided another *fractio continua*, which continues to infinity, according to a certain law, and completely removes the hope of determining the ratio of the diameter to the circumference by means of whole numbers.

§. 11.

There are other quantities in mathematics regarding which it would also be worth finding out whether they can be expressed by rational fractions or otherwise in some more tractable manner than is the case at present with their expression through decimal numbers. In particular, the number 2, 718281, 828459, 045235, 36028... can be counted among these, the hyperbolic logarithm of which is = 1.[26] This number is with respect to logarithms just what the **Ludolphian** numbers are with respect to the circle, and thus it is of **equal relevance** with respect to trigonometric and other calculations. If, therefore, one wonders why it is only the **Ludolphian** numbers that attract so much attention, then this question can only be answered partly from the history of mathematics, and partly by considering the fact that the concepts of **circle**, **square** and **quantity** are known to all alike, which cannot be said about the concept of **hyperbolic logarithms**, since this latter concept only became known through

[26] At the time the natural logarithms were also called «hyperbolic» «because of their correspondence with the quadrature of the hyperbola» (Clemm 1768, p. 432). Euler himself points this out in Euler (1748, p. 90).

infinitesimal calculus and cannot be clarified without learning this calculus. Had not most of those who have sought the quadrature of the circle encountered this block and barrier to their efforts, then most likely as many vain efforts and failed attempts would have come to light with regard to the number 2, 718281, 828459, 045235, 36028... as have come to light with regard to the **Ludolphian** numbers. However, this number cannot be expressed exactly by means of a rational fraction either. Because, if for the sake of brevity one sets this number $= e$, then one has

$$e = 1 + \cfrac{2}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cfrac{1}{26 + \text{ etc.}}}}}}}}$$

or

$$\frac{e-1}{e+1} = \cfrac{1}{2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 \text{ etc.}}}}}}$$

or

$$\frac{ee-1}{ee+1} = \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \text{ etc.}}}}}}}$$

and, in general,

$$\frac{e^x-1}{e^x+1} = \cfrac{1}{2:x + \cfrac{1}{6:x + \cfrac{1}{10:x + \cfrac{1}{14:x + \text{ etc.}}}}}$$

Since these fractions go on forever, then neither e nor e^x can be expressed exactly by a rational fraction, when x is a rational number or a fraction. I discovered, moreover, these formulae by using the method I described in the above-mentioned treatise on the **Transformation of Fractions** (§. 19ff.). However, the specific occasion to look for these formulae was provided for me by Mr **Euler's** *Analysis infinitorum*, where the expression

$$\frac{e-1}{2} = \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \text{ etc.}}}}}}$$

appears in the form of an example, calculated in numerical terms.

§. 12.

Seizing this occasion, I went further and, with respect to the arcs of the circle, I found the expression

$$\tan .v = \cfrac{1}{1:v - \cfrac{1}{3:v - \cfrac{1}{5:v - \cfrac{1}{7:v - \cfrac{1}{9:v - \text{ etc.}}}}}}$$

Several consequences can be drawn from this continued fraction with regard to the indeterminate quadrature of the circle. If one considers a whole number n, and makes $v = 1 : n$, then one obtains

$$\tan .v = \cfrac{1}{n - \cfrac{1}{3n - \cfrac{1}{5n - \cfrac{1}{7n - \cfrac{1}{9n - \cfrac{1}{11n - \text{ etc.}}}}}}}$$

Since this fraction goes on forever, it follows that, in every case where a circular arc is a *pars aliquota* of the radius, its tangent will necessarily be irrational. Because, if the tangent were rational, then this fraction could not be a continued one but would eventually have to cease. To explain this in more detail, let us consider $v = 1$ as an example. Since n also becomes $= 1$, then

$$\tan .\text{arc. } 1 = \cfrac{1}{1 - \cfrac{1}{3 - \cfrac{1}{5 - \cfrac{1}{7 - \cfrac{1}{9 - \text{ etc.}}}}}}$$

and therefore, according to the above-mentioned treatise (§. 10),[27]

	+1	+0
+1	+0	+1
-3	+1	+1
+5	-3	-2
-7	-14	-9
+9	+95	+61
-11	+841	+540
+13 etc.	-9156 etc.	+5879 etc.

And so the tangent of the arc equal to the radius is expressed through the ordered fractions

$$\frac{3}{2}, \frac{14}{9}, \frac{95}{61}, \frac{841}{540}, \frac{9156}{5879} \quad \text{etc.}$$

and expressed indeed, through every fraction in this sequence, with such a degree of progressively greater exactness, that the smaller a fraction is the less exact it is. Since this sequence of fractions is never interrupted, but continues in such a way that, having no common divisors, the denominator and numerator become larger than any given number, then the tangent of the arc equal to the radius cannot be expressed by any finite or rational fraction. This also applies to the tangents of all arcs that are $\frac{1}{n}$ part of the radius.

§. 13.

If the first fractions found here are subtracted from one another, it can be seen how quickly they approach the real value. For one has

[27] Due to the error in the numbering of the paragraphs, the original refers here to §. 9, Lambert (1766/1770, p. 163). It should also be noted that there is a mistake in the table, where instead of +5879 it should be −5879 (Lambert 1766/1770, p. 163), cf. Rudio (1892, p. 151).

$$\frac{14}{9} = \frac{3}{2} + \frac{1}{2 \cdot 9}$$
$$\frac{95}{61} = \frac{3}{2} + \frac{1}{2 \cdot 9} + \frac{1}{9 \cdot 61}$$
$$\frac{841}{540} = \frac{3}{2} + \frac{1}{2 \cdot 9} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540}$$
$$\frac{9156}{5879} = \frac{3}{2} + \frac{1}{2 \cdot 9} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540} + \frac{1}{540 \cdot 5879} \text{ etc.}$$

If one proceeds in this way, the tangent of the arc equal to the radius is expressed by an infinite series

$$\frac{3}{2} + \frac{1}{2 \cdot 9} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540} + \frac{1}{540 \cdot 5879} + \frac{1}{5879 \cdot 76887} + \text{ etc.,}$$

which converges more markedly than does any geometric series, and the sum of which is known to be irrational.

§. 14.

It thus becomes clear that not only the tangents of the arcs $\frac{1}{n}$, but in general those of all arcs $\frac{m}{n}$, which have a rational ratio to the radius, are irrational. If, for example, $v = \frac{2}{3}$, then the tangent of this arc is

$$= \cfrac{1}{3:2 - \cfrac{1}{9:2 - \cfrac{1}{15:2 - \cfrac{1}{21:2 - \text{ etc.,}}}}}$$

according to which

	1	0		
+3 : 2	0	1	=	0 : 1
− 9 : 2	+1	+3 : 2	=	2 : 3
+15 : 2	−9 : 2	−23 : 4	=	18 : 23
−21 : 2	−131 : 4	−333 : 8	=	262 : 333
+27 : 2	+2715 : 8	+6901 : 16	=	5430 : 6901
etc.	etc.	etc.		

Thus, the tangent of the arc $v = \frac{2}{3}$ is expressed through each of the fractions $\frac{2}{3}$, $\frac{18}{23}$, $\frac{262}{333}$, $\frac{5430}{6901}$ etc. and expressed indeed, through every fraction in this sequence, with such a degree of progressively greater exactness, that the smaller a fraction is the less exact it is. Since this sequence of fractions never ceases, but grows in such a way that, having no common divisors, the denominator and numerator become larger than any given number, it follows that the tangent of the arc $v = \frac{2}{3}$ is irrational. This also applies to the tangents of all arcs which are $= \frac{m}{n}$ or have a rational ratio to the radius. If one subtracts the first fractions found here from each other, one obtains for the tangent of the arc $v = \frac{2}{3}$ the series

$$\frac{2}{3} + \frac{8}{3 \cdot 23} + \frac{32}{23 \cdot 333} + \frac{128}{333 \cdot 6901} + \text{ etc.}$$

which also converges more markedly than any geometric series and has an irrational sum.

§. 15.

Since, therefore, the tangent of every rational arc is irrational, then, conversely, the arc of every rational tangent is irrational. For, if one were to assume the arc to be rational, then, contrary to the assumption, the tangent would be irrational by virtue of what was initially proven.

§. 16.

In the trigonometric tables we have a single rational tangent, namely that of 45 deg., which is equal to the radius and, therefore, $= 1$. For this reason, the arc of 45 deg. is irrational and likewise irrational, consequently, are the arcs of 90, 180 and 360 deg., or in other words, these arcs have no rational ratio to the radius of the circle.

§. 17.

From what has been said so far it is clear that no arc can at the same time have a rational ratio to the radius and to its tangent. There are, however, innumerable ways in which an arc can have a rational ratio to its tangent. But it can also be shown that, in all such cases, both the arc and its tangent are *incommensurable* with the radius. Because in the first place, by virtue of what has already been proven, it is not possible for both to have at the same time a rational ratio with respect to the radius. Let it therefore be assumed that only the tangent or the arc is rational. In the first case, the tangent would have to be *commensurable* with both the radius and the arc. And, thus, the arc would also be *commensurable* with the radius, since the sum or the difference of two rational ratios is also rational. In the other case, the arc would be *commensurable* with the tangent as well as with the radius, and thus the tangent would also have a rational ratio to the radius. Now, since, by virtue of what has been proven above, the radius, the arc and the tangent are not all at the same time *commensurable*, then both of the above cases are invalidated. Accordingly, if the arc and the tangent have a rational ratio to each other, then both are *incommensurable* with the radius.

§. 18.

I will end by briefly addressing two cases which present some plausibility with regard to the quadrature of the circle. The first is the following proposition: if one describes an arbitrary regular or irregular polygon around a circle, so that each side of the former touches the circle, then the perimeter of the polygon will thereby stand in the same relation to its content as does the circumference of the circle to its own content. I omit the proof, because it is very easy. The other case is a *phaenomenon* which occurs in the following way: if one divides 1 by 0, 7853981634..., as a fourth part of the **Ludolphian** numbers, it occurs 1 time and subtracts 0, 2146018366.... If one further divides 0, 7853981634..., which was previously the divisor, by this remainder, then it occurs 3 times and subtracts 0, 1415926536.... If one places the number 3 in front of this remainder, one gets 3, 1415926536...,[28] which are precisely the **Ludolphian** numbers.[29] I will say no more about this other than that it is a *phaenomenon* from which nothing can be concluded per se about the quadrature of the circle. Nor is it difficult to find the cause of it.

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[28] In the original this appears as 3, 1415926536 $\div$ .. (Lambert 1766/1770, p. 169).

[29] Rudio introduced the following corrections: 0, 7853981633..., 0, 1415926535... and, therefore, 3, 1415926535... (Rudio 1892, p. 155).

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Part III
Eduardo Dorrego López

Chapter 4
Introductory Remarks About the *Mémoire* (1761/1768)

> *Therefore the circumference of the circle is not to the diameter as an integer number to an integer number.*
>
> —J. H. Lambert *Mémoire*.

4.1 Introduction and Context

The historical importance of Lambert's *Mémoire* turns out evident as soon as one realizes the issues tackled by the Swiss.[1] There is little doubt that fame goes to the first part of the article, in which Lambert, showing a high level of skill with such then-recent analytic tools like continued fractions, demonstrates with unusual rigour for the 18th century standards the irrationality of π. The issue of the nature of this constant had taken a new impulse since the herculean efforts by Ludolph van Ceulen at the end of the 16th century with the use of new analytic tools and their application to some geometric problems. Authors like Gregory, Huygens, Mengoli, Leibniz or Wallis faced these issues, and in particular, the circle-squaring problem, in which π played a central role. Lambert takes up the baton of this analytic tradition —enriched by Euler with his first systematic study of continued fractions— and settles the question of its irrationality.

In any case, it may be necessary to clarify this last statement, since doubts have arisen around Lambert's proof. Such is the case of Ferdinand Rudio or Felix Klein

[1] I only intend to make some comments without going into details, since all the relevant explanations will be included in the part dedicated to the annotated translation.

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https://doi.org/10.1007/978-3-031-52223-9_4

(following the former), who refered to Lambert's proof as incomplete in contrast to authors like Alfred Pringsheim or J. W. L. Glaisher.[2] Pringsheim wrote that Lambert's work is «highly ingenious and flawlessly in essence»,[3] whereas Glaisher in his article *On Lambert's Proof of the Irrationality of* π, *and on the Irrationality of certain other Quantities*, compared his proof with that by Legendre —a proof that filled in the gaps of Lambert's according to some interpretations— in the following terms:[4]

> Although Legendre's method is quite as rigorous as that on which it is founded, still, on the whole, the demonstration of Lambert seems to afford a more striking and convincing proof of the truth of the proposition.

And he added up, after his presentation, that:

> That Lambert's proof is perfectly rigorous and places the fact of the irrationality of π beyond all doubts, is evident to every one who examines it carefully; and considering the small attention that had been paid to continued fractions previously to the time at which it was written, it cannot but be regarded as a very admirable work.

As will be seen later, the interpretation given in our analysis of Lambert's proof will show basically one single conflicting point in his demonstration; one step taken by Lambert that is not obvious and needs proof, but that would not deserve to be the source of criticism considering the epoch in which it was elaborated (1761/1768).

Although it is difficult to advance this claim with absolute certainty, it could have ocurred with Lambert and his proof what allegedly happened to Euler in connection with his proof of the irrationality of e. Some historians consider Euler's «proof» to be the statement included in the last chapter of the first volume of his *Introductio*. There,[5] Euler presented an infinite regular continued fraction for $\frac{e-1}{2}$, which ensures its irrationality, but did not provide any justification, limiting himself to comment: «This result can be confirmed by infinitesimal calculus». But what really happened is that Euler had already published a rigorous proof in a much less known work.[6]

Lambert could be in the same situation: on the one hand he published an article whose title clearly indicates the topic to be covered, and the content of which is aimed at a more general audience —this is part V of his (Lambert 1766/1770), entitled[7] *Preliminary knowledge for those who seek to square and rectify the circle*—, therefore an accessible work in which he made some claims about the nature of π; and on the other hand we have a more academic work without a direct reference in its title to the circle-squaring problem or π, and directed to a smaller audience —his (Lambert 1761/1768)— and therefore a much less known work that, on the contrary, does include a rigorous proof of the irrationality of π.

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[2] See Baltus (2003).

[3] Cantor (1908, p. 447) (translated by José Ferreirós).

[4] Glaisher (1871, p. 12).

[5] Euler (1748, p. 325).

[6] See Petrie (2009, p. 105) —who refers to a study by Ed Sandifer— for a more complete explanation. The lesser known work of Euler that I refer to is Euler (1744).

[7] *Vorläufige Kenntnisse für die, so die Quadratur und Rektifikation des Cirkuls suchen*.

An example of this interpretation could be the case of A. L. Crelle, who in his translation of Legendre's work (German 3rd edition, 1837) noted, in reference to (Lambert 1766/1770), that Lambert's proof of the irrationality of π is less rigorous than Legendre's.[8] In more recent literature, we have an example in (Beckmann, 1971, pp. 170, 171), where it is said —also making reference to (Lambert 1766/1770)— that:

> Lambert investigated certain continued fractions and proved the following theorem:
>
> *If x is a rational number other than zero, then* $\tan x$ *cannot be rational.*

adding that «Legendre, in his *Elementes de Géometrie* (1794) proved the irrationality of π more rigorously [...]».[9] Exactly in the same vein is Ebbinghaus et al. (1988, p. 149), where the authors, making reference to (Lambert 1766/1770), mention that «Lambert's proof is not completely rigorous because it lacks a lemma on the irrationality of certain continued fractions», lemma that would later be proven by Legendre.

Perhaps, we should say that Lambert is partly to blame for propagating this view, since in Lambert (1766/1770, p. 167) he expressed himself in the following terms:

> Since, therefore, the tangent of every rational arc is irrational, then, conversely, the arc of every rational tangent is irrational. For, if one were to assume the arc to be rational, then, contrary to the assumption, the tangent would be irrational **by virtue of what was initially proven**.[10]

In any case, this interpretation may partially account for this type of statements, although it should not be taken as a definitive explanation. For instance, we have already mentioned the case of Rudio, who in (Rudio 1892) seems to make explicit reference to Lambert's *Mémoire* when he talks about the fact that there is a gap in his proof, and that this gap would be eventually filled in by Legendre in his aforementioned note. Unfortunately, he does not indicate the specific place in Lambert's reasoning where he considers the problem to be at. In any case, Pringsheim in his *Ueber die ersten Beweise der Irrationalität von e und* π, comments, referring among others to Rudio and Klein, that from his point of view the interpretation according to which Legendre completed Lambert's proof is not well founded, and that Lambert proved this (and other facts[11]) «*with a rigor that is truly exceptional* for his time», bringing out the fact that Lambert included a proof of convergence whilst Legendre did not.

The general impression is that mathematicians in this period were heavily oriented towards problems and methods of calculation, general formulas and numerical approximation, while the orientation taken by Lambert in 1761/1768 was of a clearly more theoretical and/or logical tendency.[12] We are dealing in some sense with a pio-

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[8] I want to thank José Ferreirós for his comments on this respect.

[9] (Beckmann, 1971, pp. 170, 171).

[10] See Chap. 3, §. 15 (words in bold are mine).

[11] The irrationality of e^x with x a non zero rational.

[12] Concerning the importance of Lambert in the field of logic, see (Hintikka and Spade 2019), where he is claimed to have been without doubt «the greatest 18th-century logician».

neering result, a little too early for this time, falling more squarely within the general orientation of mathematics from approximately 1825 onwards. In this respect, it is insightful what A. von Braunmühl writes in Cantor (1908, pp. 447–448) after naming some authors and their connection with π:

> Lambert's contemporaries however seem either to have missed his work, or to have ignored the significance of the step that he made by establishing the knowledge of the nature of number π by means of this exact proof [...] The whole nature of Lambert's proof procedure, with its goal of absolute exactness, remained thus quite outside of the sphere of activity of their contemporaries, directed almost exclusively to the formal expansion of mathematics, and so it becomes understandable that it could be ignored.[13]

Leaving aside the irrationality of π, it is specially interesting that Lambert in this paper makes one of the first modern uses of hyperbolic functions, although the terminology he uses is not that which we currently use (he speaks of «logarithmic transcendent quantities»). After noting the similarity between the series representation of the circular and hyperbolic trigonometric functions, he looks into the reason behind that similarity, a reason that he finds in the fact that while the former parameterizes the circumference, the latter parameterizes the hyperbola.

Lastly and near the end of the work, Lambert makes the first modern distinction between algebraic and transcendental irrationals. Throughout the paper, it is possible to grasp how the modern meaning of the term «transcendent» emerges, going from representing non-finitely expressible irrational quantities, that is to say, irrational quantities that cannot be finitely expressed by means of usual algebraic operations (addition, subtraction, multiplication, division and extraction of roots), to quantities that are not roots of algebraic equations. In any case, this use did not become standard until new results in the field of algebra and number theory —Abel-Ruffini's theorem and Liouville's theorem— motivated the change from the old theoretical framework («to be expressible») to the new one («to be root»). The final part of the work culminates concretely and notably with the conjecture of the transcendence of π and the impossibility of squaring the circle.

In general, the *Mémoire* is not a self-contained work, in the sense that it does not allow a gentle and easy reading. It is enough to make this clear by bringing up what Legendre says in Note VI of his *Elements of Geometry* —entitled *Where it is shown that the ratio of the circumference to the diameter and its square are irrational numbers*, and in which he gives a new, much shorter and simpler proof of this fact— about (only) the part dedicated to the proof of the irrationality of π:

> We already know one proof of this proposition that has been given by Lambert in the Memoirs of Berlin, year 1761; but, as this proof is long and difficult to follow, we have tried to shorten and simplify it.[14]

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[13] I thank José Ferreirós for the translation.

[14] Legendre (1794, p. 296). In the second edition the demonstration is included in Note V and from the fourth in Note IV (I could not consult the third edition). The comment to Lambert from the fourth edition is reduced to a brief footnote:

> This proposition was first demostrated by Lambert, in the Memoirs of Berlin, *anno* 1761.

That is why a guide is useful and even necessary for those who want to read and understand this important work without falling into the temptation to abandon and go directly to simpler demonstrations of irrationality, such as the one by Legendre, or the more modern by Ivan Niven; needless to say for those who wants to delve into the reading of the entire article. A summary of what one could find —and indeed what the author of these lines found— in the search for such a guide would be the following.

To begin with, this is the first definitive English translation to be published.[15] The works that are constantly referred to are commentaries or translations of certain parts of the *Mémoire*. What there is is an annotation in French of the complete work in its original language; it is (Speiser 1946–1948). This work by Andreas Speiser, who edited Lambert's mathematical work, is the classic work to which historians often referred —for example (Serfati, 1992, p. 75)— but the ten footnotes that accompany the edition of the *Mémoire* are far from solving all the obstacles one would face when reading it. It has been decided to also include Speiser's annotations (see Appendix C) due to the classic nature of his work, so that in any case the reader will be able to judge for himself.[16] What is undeniable is the meticulousness of his analysis, as Speiser corrects every single of Lambert's errors or misprints, even when some of them are not easy to locate. Sometimes he includes the correction in a footnote, other times he introduces it directly in the main body of the text, including the original in a footnote, and some other times he corrects the text without mentioning it.

Further comments on this work,[17] especially regarding the demonstration of the irrationality of π, can be found in (Struik, 1969, pp. 369–374) and (Berggren, 1997, pp. 369–374), although both of them contain the same material: only points 37–51 of the *Mémoire* (with no extensive analysis). The first 37 points and points from 51 to 91 are missing and they are not of little interest as we have already mentioned briefly.

One of the most relevant works that addresses Lambert's *Mémoire* is (Serfati, 1992, pp. 62–83), which, however, does not focus on technical details but offers a more global approach to the work, in particular about the points where Lambert touches on issues of irrationality and transcendence; in his more recent work ((Serfati, 2018, pp. 179–184)), one finds basically the same material. On the other hand, for

[15] In the course of writing this book, on which we began work in late 2019 and which underwent a blind peer review process, an English translation of (Lambert 1761/1768) by Denis Roegel was published online at https://hal.archives-ouvertes.fr/hal-02984214. The translation by Roegel, which I did not use for the preparation of my own translation, was labelled by him as follows: "this is a preliminary draft[;] please check for the final version". To our knowledge, however, such a final version has not yet been published, hence I describe my translation as the first definitive one to be published.

[16] Speiser's annotations will be indicated throught the translation of the *Mémoire* by means of footnotes as follows: «See the note by A.S. in Appendix C».

[17] I have to say that this summary does not show my journey in chronological order, since there are, as the reader will know, sources that are faster and easier to consult than others. For example the aforementioned work by Adreas Speiser was the last one that I have been able to analyze, long after I had almost completely prepared the translation with the annotations.

more technical details readers are referred to the important contribution by (Baltus 2003), who addresses therein the main problematic part of Lambert's proof.

On the other hand, Martin Mattmüller and Franz Lemmermeyer in their edition of the correspondence between Euler and Goldbach, mention an article by Bruce J. Petrie (Petrie 2009) «for a modern exposition of Lambert's proof».[18] Certainly this is a helpful work containing detailed explanations —based in part (Struik 1969)—, but for most of Lambert's proof (points 1–37) Petrie simply refers to (Chrystal, 1906, pp. 517–523), who does not follow it literally but makes a modern and very technical approach, and to the work by (Brezinski, 1991, 109–111), who only makes a few comments on the three pages he devotes to Lambert's proof. In fact, Brezinski refers to (Struik 1969) for an English translation.

A bit out of the «most typical», and regarding Lambert's treatment of the hyperbolic functions contained in the *Mémoire* (Lambert is a pioneer in this), there is a valuable work —(Barnett 2004)— which analyzes the role played by Lambert in this issue, although there are things that the author leaves behind and that Lambert does include and use (such as the curious concept of «prime tangent»). Also in (Juhel 2009) one can find a useful analysis of both this issue and the parts of the *Mémoire* devoted to the irrationality of π and the conjecture launched by Lambert about the transcendence of π.

Although this brief summary does not cover all the material on the subject, what has been said should help to get an idea of the gaps surrounding this important work and justify the suitability of the publication of an annotated translation of *Mémoire sur quelques propriétés remarquables des quantités transcendentes circulares et logarithmiques* (1761/1768).

By the way, this double dating may require an explanation. The years 1761/1768 probably refer to the delivery period of the different works included in this issue of the Memoirs of the Berlin Academy. In the particular case of Lambert's *Mémoire*, we know that it was written a few months after his (Lambert 1766/1770), something he did in 1766.[19] Karl Bopp indicates that Lambert wrote the *Mémoire* in 1767, and that it was published in 1768, as the Academy's own volume indicates. In fact, Lambert made an annotation in his *Monatsbuch*, dated July 1767, which begins as follows:

> Sur une proprieté remarquable des quantités transcendentes
> circulaires et Logarithmiques. Diss[ertatio] acad[emica].

In addition, the minutes reported on the Academy had established the holding of weekly plenary sessions every Thursday. They were attended by ordinary members and works pending publication were read out. In the plenary session of Thursday, September 17, 1767, the reading of Lambert's *Mémoire* is registered.[20]

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[18] Lemmermeyer and Mattmuller (2015, p. 55 note 65).

[19] See (Lambert 1766/1770, p. [II]).

[20] The interested reader can consult the aforementioned minutes on the Berlin Academy of Science website. The reference to the reading of this work also appears in the lower left corner of the first page of the *Mémoire*: «Read in 1767». Concerning the aforementioned dating of Lambert's work, (Rudio 1892) warns us that, although many people repeat it, the data 1761 as the publication date is

4.2 Outline

What follows is a brief outline of the *Mémoire*. It is just intended to provide a general idea of Lambert's paper.

- **§. 1.–§. 3. (pp. 265–267)**

  $*$ Brief historical introduction on π.
  $*$ Intuitive explanation of the reasons to expect the irrationality of π.
  $*$ What is intended to demonstrate:

$$[v \in \mathbb{Q} \Longrightarrow \tan v \not\in \mathbb{Q}] \tag{4.1}$$

  and how to do it: by using Euclid's Algorithm for the calculation of the greatest common divisor.

- **§. 4.–§. 15. (pp. 267–275)**

  $*$ Continued fraction expression for the tangent.
  $*$ Proof by induction of said expression.
  $*$ Particular cases in which (4.1) can be proved.

- **§. 16.–§. 30. (pp. 276–286)**

  $*$ Search and proof by induction for the general term by recurrence of the succession of the convergents $\{\frac{p_n}{q_n}\}$ of the continued fraction of $\tan v$.
  $*$ Search and proof by induction —on the basis on this general term by recurrence— of the general term of this very sequence dependent only on n.
  $*$ Proof that this sequence actually converges to $\tan v$.

- **§. 31.–§. 51. (pp. 286–297)**

  $*$ Series expression for the tangent from the convergents.
  $*$ Proof by *Reductio Ad Absurdum* of (4.1).
  $*$ As an immediate consequence, $\pi \not\in \mathbb{Q}$ since $\tan \frac{\pi}{4} = 0 \in \mathbb{Q}$.

- **§. 52.–§. 71. (pp. 297–304)**

  $*$ Results that motivate the introduction of the concept of «prime tangent».
  $*$ The case of $\tan 45°$.
  $*$ Results on prime tangents.

wrong. The relevant parts of the *Monatsbuch* in this regard are (Bokhove and Emmel, 2020, pp. 112 (note 527), 164 (note 733), 169 (note 763), 172 (note 773)). I would like to clarify that I have been able to access this work thanks to the kindness of Armin Emmel, who sent me the parts related to my investigation in a totally disinterested way. Likewise, I thank José Ferreirés for the translation of these parts.

* The similarity with the cosine (completely analogous to the tangent, we would have the concept of prime cosine with similar results) and the difference with the sine (not in this case).

- **§. 72.–§. 88. (pp. 304–320)**

  * Search for fractions that approximate the continued fraction both by default and by excess. Continued fraction expression of the cotangent.
  * Similarity between the infinite series of the «circular transcendent quantities» (sine and cosine) and the «logarithmic transcendent quantities» (hyperbolic sine and cosine). Continued fraction expression of some rational expressions of e^x. New irrationality result:

$$x \in \mathbb{Q} \Longrightarrow e^x \not\in \mathbb{Q}$$

  * Link between the «logarithmic transcendent quantities» and the hyperbola equilateral, in the same way that this link occurs between the «circular transcendent quantities» and the circumference.
  * New irrationality results: hyperbolic tangent and natural logarithms.
  * What appears to be an affirmation of the transcendence —no longer in the classical sense of the term, but in its modern sense— of the number e. Irrationality results for hyperbolic logarithms.
  * How the concept of «prime tangent» applies equally to the hyperbolic case.
  * One last look at the analogy between «circular transcendent quantities» and «logarithmic transcendent quantities».

- **§. 89.–§. 91. (pp. 320–322)**

  * First modern differentiation between irrational in terms of algebraic and transcendental.
  * Conjecture of the transcendence of the «logarithmic transcendent quantities» and the «circular transcendent quantities».
  * Conjecture of the significance of π and therefore of the impossibility of squaring the circle.

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Chapter 5
An Annotated Translation of Lambert's *Mémoire* (1761/1768)

The way I have proved this [the irrationality of π] can be extended to the point that circular and logarithmic quantities cannot be roots of rational equations.

—J. H. Lambert, *letter to Holland.*

MEMOIRE

ON

SOME REMARKABLE PROPERTIES OF
CIRCULAR AND LOGARITHMIC TRANSCENDENTAL QUANTITIES[1]

BY M. LAMBERT.[2]

§. I.

Proving that the diameter of the circle is not to its circumference as an integer number to an integer number is something that will hardly surprise geometers. We know *Ludolph*'s numbers, the ratios found by *Archimedes*, by *Metius* etc. as well as

[1] Transcendental in the Eulerian-Leibnizian sense, although it will be precisely in this work where this term will adquire its modern meaning (§. 89–§. 91). On the other hand, the term «logarithmic quantities» refers to the hyperbolic ones given the connection between the hyperbola and the logarithmic function (we will stress on this later in this chapter).

[2] «Read in 1767.
Mém. of the *Acad.* Tom. XVII».

E. Dorrego López and E. Fuentes Guillén, *Irrationality, Transcendence and the Circle-Squaring Problem*, Logic, Epistemology, and the Unity of Science 58,
https://doi.org/10.1007/978-3-031-52223-9_5

a large number of infinite series, all of which refer to the quadrature of the circle. And if the sum of these series is a rational quantity, we must naturally conclude that it will be either an integer number or a very simple fraction. Since, if it were a very composed fraction, what reason would there be? Why this one and not any other?[3] It is in this way, for example, that the sum of the series

$$\frac{2}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}+\frac{2}{7\cdot 9}+\text{\&c.}$$

[3] Based on this premise and on the fact that some of the approximations found for the area of the circle, far from being simple, are increasingly complex ($\frac{11}{14}$ and $\frac{355}{452}$), Lambert concludes at the end of this point that this area must be an irrational quantity: what reason would there be for this quantity to be rational being it such a composed fraction? Although it seems clear that he is intending to draw attention to a certain fact that should lead anyone to trust on its irrationality, certainly this «proof by simplicity» (I take the term from Serfati (2018, p. 180)) seems too vague to be taken seriously. But, more surprisingly, in §. 2, although he comments that a problem of this kind as it is the squaring of the circle cannot be left to a reasoning of these characteristics, «there are nevertheless cases in which more is not required». The mandatory question is: in which cases could a reasoning like this be enough? It seems the opposite of what is demanded for a mathematical justification. What is clear is that this kind of reasoning must be a reflection of a more or less generalized way of thinking and not an isolated case. In fact, this type of proofs by simplicity can be found in the works of other authors of the time. For example, in Bullynck (2009, p. 147) Wolfram is said to have falsely assumed «that every root of an equation can be made rational through exponentiation», which would mean that otherwise we would have a transcendental quantity. This could actually be just a new application of this principle by simplicity: if $\alpha^n \notin \mathbb{Q}$ for every $n \in \mathbb{N}$, we would be telling that α is not the root of any of the following equations: $x^n + q = 0$ (that is, it has all the ballots to be transcendental); if it is not the root of this type of equations, using this principle, one could conclude that it is not the root of any. Euler (1785, p. 8) clearly uses a very similar principle to that used by Lambert, when, after showing that $\pi \neq a\sqrt{2} + b\sqrt{3}$, he concludes that π cannot be expressed by radicals as follows: «I will not continue these operations further, since if an exact relation were to be given, without doubt it would not be so complicated». Furthermore, we can find examples that would sound even stranger, in which results are established with the help of certain non-mathematical reasoning. The case of Leibniz (see Español et al. (2008, pp. 185–186)), someone who by the way influenced Lambert, at least through Wolff, is clear when in a letter to the latter he defends the value $\frac{1}{2}$ as result of the operation $1-1+1-1+1-1+\cdots$ since: although if, on the one hand, by truncating that sum into an even number of addends it yields 0, and, on the other hand, into an odd number of addends it yields 1, by continuing the sum to infinity the difference between even and odd is blurred, and so these two values appears with the same average, and therefore the resultant value of that sum should be $\frac{0+1}{2} = \frac{1}{2}$. According to Leibniz:

> Although this kind of argumentation can be seen as more metaphysical than mathematical, it is nevertheless firm: and, on the other hand, the use of the rules of true metaphysics (which goes beyond the nomenclature of the terms used) in Mathematics, in Analysis, in Geometry itself, is more frequent than people think.

This type of reasoning was also adopted by several of the Bernoullis and criticized by Laplace as late as 1812, which shows that it was known and discussed Español et al. (2008, pp. 186–187), Klein (1983, pp. 307–308). The surprise that one can fell when reading things like these, reminds us better than anything the almost 300 years that separate us from these mathematicians.

is equal to unity,[4] which of all rational quantities is the simplest. But, alternatively, by omitting the terms 2, 4, 6, 8 &c., the sum of the others

$$\frac{2}{1\cdot 3}+\frac{2}{5\cdot 7}+\frac{2}{9\cdot 11}+\frac{2}{13\cdot 15}+\&\text{c.}$$

gives the area of the circle when the diameter is $= 1$.[5] It therefore seems that, if this sum were rational, it should also be able to be expressed by a very simple fraction, such as $\frac{3}{4}$ or $\frac{4}{5}$ &c. Indeed, being the diameter $= 1$, the radius $= \frac{1}{2}$, the square of the radius $= \frac{1}{4}$, it is clear that these expressions, being also simple, do not pose an obstacle.[6] And since it is about the entire circle, which makes a kind of unity, & not just about some Sector, which due to its nature would require very large fractions,[7] it is clear that in this respect we should not expect a very composed fraction either. But since, after the fraction $\frac{11}{14}$ found by *Archimedes*, which gives only an approximation, we move on to that of *Metius*, $\frac{355}{452}$, which is not exact either, & in which the numbers are considerably larger, we must conclude that the sum of this series, far from being equal to a simple fraction, is an irrational quantity.

§. 2. As vague as this reasoning is, there are nevertheless cases in which nothing more is required. But these are not the cases for squaring the circle. Most of those who seek it, do so with an ardor which sometimes leads them to cast doubt on the most fundamental & well-established truths of geometry. Could it be believed that they would be satisfied by what I just said? Something else is needed. And if the issue is to demonstrate that, in fact, the diameter is not to the circumference as an

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[4] Since:

$$\frac{2}{1\cdot 3}+\frac{2}{3\cdot 5}+\frac{2}{5\cdot 7}+\frac{2}{7\cdot 9}+\cdots=\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots=1$$

(Serfati 2018, p. 180 note 32).

[5] Leibniz had found in 1673 —and independently J. Gregory and Nilakantha (see Berggren (1997, pp. 92, 93, 97 note 4))— from the infinite series for the arctangent, that the area of the circle of radius one can be expressed by the series:

$$\mathcal{A}_C=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots$$

Grouping the terms two by two:

$$\begin{aligned}\mathcal{A}_C=\frac{\pi}{4}&=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots\\&=\frac{2}{1\cdot 3}+\frac{2}{5\cdot 7}+\frac{2}{9\cdot 11}+\frac{2}{13\cdot 15}+\cdots\end{aligned}$$

we obtain precisely the expression presented by Lambert.

[6] They do not pose any obstacle for the value of the area $\frac{1}{4}\cdot\pi$ to be a *fraction fort simple*, because $\frac{1}{4}$ is itself a *fraction fort simple*; therefore the problem lies with π.

[7] If the area of the circle were a fraction, a sector of the circle would be a fraction of the fraction of the area, and therefore a *fraction fort grand*.

integer number to an integer number, this demonstration must be as firm as any geometric demonstration. And with all this I say again that geometers will not be surprised by it. They must have long been used to expecting nothing else. But here is what will deserve more attention, & what will be a good part of this Memoir. The issue is to show that *every time an arc of a circle is commensurable to the radius, the tangent of that arc is incommensurable to it*; & that conversely, *every commensurable tangent is not the tangent of a commensurable arc.*[8] Here is what to be a little more surprised about. This statement would seem to admit an infinity of exceptions, & admits none. It remains to be seen to what extent transcendental circular quantities are transcendental, & beyond all commensurability.[9] As the demonstration that I am going to give requires all the geometric rigor, & in addition it will be a network of other theorems, which will demand to be demonstrated with equal rigor, these reasons will excuse me when I do not hurry to reach the end, or when on the way I stop with whatever is remarkable.

§. 3. Let then *be given any arc of a circle, but commensurable to the radius:* & the aim is to discover, *will this arc of a circle be at the same time commensurable to its tangent or not?* Let us imagine for this effect a fraction such that its numerator is equal to the arc of the proposed circle, & that its denominator is equal to the tangent of this arc. It is clear that, whatever the way in which this arc & tangent are expressed, this fraction must be equal to another fraction, in which the numerator & the denominator will be integers, provided the proposed arc of the circle be commensurable to the tangent. It is also clear that this second fraction must be able to be deduced from the first, by the same method that one uses in arithmetic to reduce a fraction to its lowest denominator. This method is known since *Euclid*, who uses it in the 2nd prop. of his 7th book,[10] I will not stop to prove it again. But it should be noted that, while *Euclid* only applies it to integer & rational numbers, will I have to use another method when it comes to applying it to quantities of which it is unknown whether they are rational or not? Hence, here is the procedure that will suit the case in question.

§. 4. Let the radius be $= 1$, and any given arc of a circle $= v$. And we shall have the two well-known infinite series[11]

[8] What he proposes to demonstrate is therefore that:

$$\text{If} \quad v \in \mathbb{Q} \Rightarrow \tan v \notin \mathbb{Q}$$

[9] As Serfati (2018, p. 181) writes, the term «transcendent» here «has no precise technical meaning [...]; it simply means that the quantities involved are irrational in an extraordinary way, beyond any standard». The author makes the same comment in reference to another expression by Lambert elsewhere in the text (specifically in §. 81 concerning e), although in that case —as will be duly explained— a modern meaning of the term seems to be already glimpsed.

[10] «*Given two numbers not prime to one another, to find their greatest common measure*» Heath II (1908, p. 298).

[11] Lambert does not use the modern notation for the factorial of a number to represent successive products, which was introduced later, in 1808, by the also Alsatian Cristian Kramp in his *Éléments d'arithmétique universelle* (*Elements of universal arithmetic*). This is certainly a good example of the simplifying power of some notations.

$$\sin v = v - \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^7 + \&\text{c.}$$
$$\cos v = 1 - \frac{1}{2}v^2 + \frac{1}{2\cdot 3\cdot 4}v^4 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}v^6 + \&\text{c.}$$

Since in what follows I will give two series for the hyperbola which differ from these two only in that all the signs are positive, I will postpone proving the law of progression of these series until then, & I will only prove it so as not to omit anything that is required by geometric rigor.[12] It is enough then to have warned the Readers in advance.

§. 5. Now since

$$\tan v = \frac{\sin v}{\cos v},$$

we shall have, substituting these two series, the fraction

$$\tan v = \frac{v - \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 - \&\text{c.}}{1 - \frac{1}{2}v^2 + \frac{1}{2\cdot 3\cdot 4}v^4 - \&\text{c.}}$$

I will put for the sake of brevity

$$\tan v = \frac{A}{B},$$

in such a way that

$$A = \sin v,$$
$$B = \cos v.$$

Here is now the procedure that *Euclid* prescribes.

§. 6. Let us divide B by A; be the quotient $= Q'$, the residue $= R'$.
Let us divide A by R'; be the quotient $= Q''$, the residue $= R''$.
Let us divide R' by R''; be the quotient $= Q'''$, the residue $= R'''$.
Let us divide R'' by R'''; be the quotient $= Q^{IV}$, the residue $= R^{IV}$. &c., so that by continuing these divisions, we find successively

[12] He refers to the series expansion for the hyperbolic sine and cosine obtained in this very work. The similarity between both expressions, along with other considerations, leads him to wonder about the underlying connection between them, something that, as he himself says (§. 74), had previously been noted by Mr. de Foncenex (1759). This connection resides in the fact that in the same way that circular trigonometric functions parameterize the circumference, hyperbolic trigonometric functions parameterize the equilateral hyperbola. Lambert seeks for the functions that parameterize the hyperbola obtaining the hyperbolic cosine and sine (see Barnett (2004) for more details).

the quotients $Q', \ Q'', \ Q''' \ldots\ldots. Q^n, \ Q^{n+1}, \ Q^{n+2} \ldots . \&c.$
the residues $R', \ R'', \ R''' \ldots\ldots. R^n, \ R^{n+1}, \ R^{n+2} \ldots .$ &c.

& it goes without saying that the exponents $n, \ n+1, \ n+2$ &c. are only used to indicate the quotient or residue with which are indicated.[13] With this as a basis, here is what we seek to prove.

§. 7. First of all, *not only that the division can be continued forever, but that the quotients will follow a very simple law that yields*

$$\begin{aligned} Q' &= +1 : v, \\ Q'' &= -3 : v, \\ Q''' &= +5 : v, \\ Q^{IV} &= -7 : v, \quad \&\text{c.} \end{aligned}$$

& in general

$$Q^n = \pm (2n-1) : v,$$

where the sign + is for the even n exponent,[14] *the sign − is for the odd n exponent,*[15] *& that in this way we will have for the tangent expressed by means of the arc the very simple continued fraction*

$$\tan v = \cfrac{1}{1:v - \cfrac{1}{3:v - \cfrac{1}{5:v - \cfrac{1}{7:v - \cfrac{1}{9:v - \&\text{c.}}}}}}$$

[13] These divisions will finally lead to a continued fraction expansion for the tangent:

$$\frac{A}{B} = \frac{1}{\frac{B}{A}} = \frac{1}{Q' + \frac{R'}{A}} = \frac{1}{Q' + \frac{1}{\frac{A}{R'}}} = \frac{1}{Q' + \frac{1}{Q'' + \frac{R''}{R'}}} = \frac{1}{Q' + \frac{1}{Q'' + \frac{1}{\frac{R'}{R''}}}} = \cdots$$

Note that the first systematic study of continued fractions was undertaken by Euler in 1737 in Euler (1744) (written in 1737 and published in 1744 in the Proceedings of the National Academy of St. Perershurg), so the tools that Lambert skillfully uses are very novel.

[14] See the note by A.S. in Appendix C.

[15] See the note by A.S. in Appendix C. Let us advance that there is a clear error at this point: the sign + corresponds to the odd exponent, and the sign − to the even exponent.

§. 8. Second,[16] *that the residue* R', R'', R''' &c. *will be expressed by the following series, where the laws of progression are also very simple:*

$$R' = -\frac{2}{2\cdot 3}v^2 + \frac{4}{2\cdot 3\cdot 4\cdot 5}v^4 - \frac{6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^6 + \&c.$$

$$R'' = -\frac{2\cdot 4}{2\cdot 3\cdot 4\cdot 5}v^3 + \frac{4\cdot 6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^5 - \frac{6\cdot 8}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}v^7 + \&c.$$

$$R''' = +\frac{2\cdot 4\cdot 6}{2\cdot\cdots\cdot 7}v^4 - \frac{4\cdot 6\cdot 8}{2\cdot\cdots\cdots\cdot 9}v^6 + \frac{6\cdot 8\cdot 10}{2\cdot\cdots\cdot 11}v^8 - \&c.$$

$$R^{IV} = +\frac{2\cdot 4\cdot 6\cdot 8}{2\cdot\cdots\cdot 9}v^5 - \frac{4\cdot 6\cdot 8\cdot 10}{2\cdot\cdots\cdots\cdot 11}v^7 + \frac{6\cdot 8\cdot 10\cdot 12}{2\cdot\cdots\cdot 13}v^9 - \&c.$$

&c.

so that the signs of the first terms will change following the quaternary order $- - + +$, & *that in general we shall have*

[16] See the note by A.S. in Appendix C. Note the following observation about the general expressions for the residues, which Lambert includes at this point. Take for example the expression for $\pm R^n$:

$$\pm R^n = -\frac{2^n(1\cdot 2\cdot\cdots\cdot n)}{1\cdot 2\cdots(2n+1)}v^{n+1} + \frac{2^{n+1}(1\cdot 2\cdot\cdots\cdot(n+1))}{1\cdot 2\cdot\cdots\cdots\cdot(2n+3)}v^{n+3} - \&c.$$

To begin with, one might think, taking into account the « &c.», that the successive products that appear in the numerators would follow the sequence $n!$, $(n+1)!$, $(n+2)!$, $(n+3)!$, …, but this is not the case. If one analyzes the numerators of the first three addends in the terms R', R'' *y* R''':

	First addend	Second addend	Third addend
R'	$2 = 2^1\cdot 1!$	$4 = 2^1\cdot 2!$	$6 = 2^1\cdot 3$
R''	$2\cdot 4 = 2^2\cdot 2!$	$4\cdot 6 = 2^2\cdot 3!$	$6\cdot 8 = 2^2\cdot(3\cdot 4)$
R'''	$2\cdot 4\cdot 6 = 2^3\cdot 3!$	$4\cdot 6\cdot 8 = 2^3\cdot 4!$	$6\cdot 8\cdot 10 = 2^3\cdot(3\cdot 4\cdot 5)$

one sees that from the third column onwards the factorials no longer appear, since the multiplicands are reduced at each step, something that also happens in the second column where what is removed is the factor 1 and this does not affect multiplicatively (here therefore the «&c.» might be a little bit misleading); this is also revealed later in the text. It should also be noted that if one gives n the value 1, the resulting expression, R', does not match the one given by Lambert. This is because the powers of 2 should not change their exponent when changing the addend; that is, they should be:

$$\pm R^n = -\frac{2^n(1\cdot 2\cdot\cdots\cdot n)}{1\cdot 2\cdots(2n+1)}v^{n+1} + \frac{2^n(1\cdot 2\cdot\cdots\cdot(n+1))}{1\cdot 2\cdot\cdots\cdots\cdot(2n+3)}v^{n+3} - \&c.$$

$$\pm R^{n+1} = -\frac{2^{n+1}(1\cdot 2\cdots(n+1))}{1\cdot 2\cdot\cdots\cdot(2n+3)}v^{n+2} + \frac{2^{n+1}(1\cdot 2\cdot\cdot(n+2))}{1\cdot 2\cdot\cdots\cdot(2n+5)}v^{n+4} - \&c.$$

$$\mp R^{n+2} = +\frac{2^{n+2}(1\cdot 2\cdots(n+2))}{1\cdot 2\cdot\cdots\cdot(2n+5)}v^{n+3} - \frac{2^{n+2}(1\cdot 2\cdots(n+3))}{1\cdot 2\cdot\cdots\cdot(2n+7)}v^{n+5} + \&c.$$

$$\pm R^n = -\frac{2^n(1\cdot 2\cdot\cdots\cdot n)}{1\cdot 2\cdots(2n+1)}v^{n+1} + \frac{2^{n+1}(1\cdot 2\cdot\cdots\cdot(n+1))}{1\cdot 2\cdot\cdots\cdots(2n+3)}v^{n+3} - \&c.$$

$$\pm R^{n+1} = -\frac{2^{n+1}(1\cdot 2\cdots(n+1))}{1\cdot 2\cdot\cdots\cdot(2n+3)}v^{n+2} + \frac{2^{n+2}(1\cdot 2\cdot\cdot(n+2))}{1\cdot 2\cdot\cdots\cdot(2n+5)}v^{n+4} - \&c.$$

$$\mp R^{n+2} = +\frac{2^{n+1}(1\cdot 2\cdots(n+2))}{1\cdot 2\cdot\cdots\cdot(2n+5)}v^{n+3}\frac{2^{n+3}(1\cdot 2\cdots(n+3)}{1\cdot 2\cdot\cdots\cdot(2n+7)}v^{n+5} + \&c.$$

§. 9. Now, in order to give all the possible brevity to the demonstration of these theorems,[17] let us consider that each residue R^{n+2} is found by dividing the antepenultimate R^n by the residue R^{n+1}, which immediately precedes it. This consideration makes possible to divide the proof into two parts. In the first we shall see that; *if two residues R^n, R^{n+1}, which follow each other immediately, have the form that I have given to them, the residue R^{n+2}, which immediately follows, will have the same form*. Once this has been proven, all that remains is to show, in the second part of the proof, *that the form of the first two residues is that which they should have*. Since, in this way, it is evident that the form of all the following is established as by itself.[18]

§, 10. Let us then start by dividing the first term of the residue R^n by the first term of the residue R^{n+1}, so as to obtain the quotient[19]

$$Q^{n+2} = \frac{2^n(1\cdot 2\cdot 3\cdot\cdots\cdot n)}{1\cdot 2\cdot 3\cdots(2n+1)}v^{n+1} : \frac{2^{n+1}(1\cdot 2\cdot 3\cdots(n+1))}{1\cdot 2\cdot 3\cdot\cdots\cdot(2n+3)}v^{n+2}$$

$$= 1 : \frac{2(n+1)v}{(2n+2)\cdot(2n+3)} = (2n+3) : v.$$

And it is clear that, the residue R^{n+1} being multiplied by this quotient

$$Q^{n+2} = (2n+3) : v,$$

& the product being subtracted from the residue R^n, the residue R^{n+2} must remain.[20]

§. 11. But in order not to have to do this operation for each term separately & thereby restrict ourselves to a simple induction, let us take the general term of each series that express the residues R^n, R^{n+1}, R^{n+2}, so that by taking the m-th term of

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[17] He will give the proof of these theorems in §. 10., §. 11., §. 12. y §. 13.

[18] What he will do, therefore, is to use the induction method, a method that will be very present throughout this work (Lambert first verifies the induction step, and then the base case).

[19] Since R^n and R^{n+1}, which is what we start from, are respectively the dividend and divisor in the n-th step, in order to obtain the residue R^{n+2} we must first to find the quotient Q^{n+2}, which is found, as in any other division between polynomials, by dividing dividend and divisor. What this reveals, as Lambert will comment in §. 14, is that the demonstration by induction of the expressions for the residues, already provides the demonstration of the expressions for the quotients.

[20] Due to the fact that $R^{n+2} = R^n - R^{n+1}\cdot Q^{n+2}$, as in any division. Lambert will now prove that by doing this operation (term by term) we effectively obtain the expected expression for R^{n+2}.

the residues R^n, R^{n+1}, we shall have the $(m-1)$-th term of the residue[21] R^{n+2}. This having been noted, the terms will be[22]

$$\pm r^n = -\frac{2^{n+m-1}(m \cdot (m+1) \cdot (m+2) \cdot \cdots \cdot (n+m-1) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2n+2m-1)}$$

$$\pm r^{n+1} = -\frac{2^{n+m} \cdot (m \cdot (m+1)(m+2) \cdot \cdots \cdot (n+m) v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2n+2m+1)}$$

$$\pm r^{n+2} = -\frac{2^{n+m} \cdot ((m-1) \cdot m \cdot (m+1) \cdot \cdots \cdot (n+m) \cdot v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2n+m+1)}$$

Now, since it must be

[21] This is because when dividing the first term by the first term —pay attention for example to the beginning of the division when dividing $\cos v$ by $\sin v$ searching for the first quotient Q' and the first residue R'— no residue is generated, but rather the quotient is obtained directly ($\frac{1}{v}$ in this case):

$$B = \cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \cdots \pm \frac{v^m}{m!} \mp \cdots$$

$$A = \sin v = v - \frac{v^3}{3!} + \frac{v^5}{5!} - \frac{v^7}{7!} + \cdots \pm \frac{v^{m+1}}{(m+1)!} \mp \cdots$$

When dividing the second term by the second term, we do obtain a residue which would in fact be the beginning of the series R', that is to say, the first addend or term (let us put R'_1):

$$R'_1 = -\frac{v^2}{2!} - \frac{1}{v} \cdot \frac{-v^3}{3!} = \frac{-2v^2}{3!}$$

When dividing the third by the third, a residue is also generated, in this case the second addend of the series:

$$R'_2 = \frac{v^4}{4!} - \frac{1}{v} \cdot \frac{v^5}{5!} = \frac{4v^4}{5!}$$

and so on.

[22] See the note by A.S. in Appendix C. In addition to some errors in the omission of parentheses and in the last multiplicand of the denominator of the last expression, it should be noted again that the exponents should not vary as a function of m. Bearing that in mind, the expressions should be written as follows:

$$\pm r^n = -\frac{2^n(m \cdot (m+1) \cdot (m+2) \cdot \cdots \cdot (n+m-1)) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2n+2m-1)}$$

$$\pm r^{n+1} = -\frac{2^{n+1} \cdot (m \cdot (m+1)(m+2) \cdot \cdots \cdot (n+m)) v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2n+2m+1)}$$

$$\pm r^{n+2} = -\frac{2^{n+2} \cdot ((m-1) \cdot m \cdot (m+1) \cdot \cdots \cdot (n+m)) \cdot v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (2n+2m+1)}$$

$$r^n - r^{n+1} \cdot (2n+3) : v = r^{n+2},$$

& that in fact it is[23]

$$r^n - r^{n+1}(2n+3) : v = -\frac{2^{n+m-1} \cdot (m \cdot \cdots \cdot (n+m-1) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n+2m-1)}$$

$$+ \frac{2^{n+m}(m \cdot \cdots \cdot (n+m) v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n+2m+1)} \cdot \frac{2n+3}{v}$$

$$= \frac{2^{n+m-1} \cdot (m \cdots (n+m-1))}{1 \cdot 2 \cdot \cdots \cdot (2m+2m-2)} v^{n+2m-1} \cdot (-1 + \frac{2 \cdot (n+m) \cdot (2n+3)}{(2n+2m) \cdot (2n+2m+1)}$$

$$= -\frac{2^{n+m-1}(m \cdots (n+m-1))}{1 \cdot 2 \cdot \cdots \cdot (2n+2m-2)} v^{n+2m-1} \cdot \frac{(2m-2) \cdot (2n+2m)}{(2n+2m) \cdot (2n+m+1)}$$

$$= -\frac{2^{n+m} \cdot ((m-1) \cdot m(m+1) \cdot \cdots \cdot (n+m) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n+m+1)},$$

[23] Again we must take into account the problem in the powers of two, in addition to several misprints. Once this has been corrected, the result would be:

$$r^n - r^{n+1} \cdot (2n+3) = r^{n+2} = -\frac{2^n \cdot (m \cdot \cdots \cdot (n+m-1)) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n+2m-1)}$$

$$+ \frac{2^{n+1}(m \cdot \cdots \cdot (n+m)) v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n+2m+1)} \cdot \frac{2n+3}{v}$$

$$= \frac{2^n \cdot (m \cdots (n+m-1))}{1 \cdot 2 \cdot \cdots \cdot (2n+2m-1)} v^{n+2m-1} \cdot (-1 + \frac{2 \cdot (n+m) \cdot (2n+3)}{(2n+2m) \cdot (2n+2m+1)})$$

$$= -\frac{2^n(m \cdots (n+m-1))}{1 \cdot 2 \cdot \cdots \cdot (2n+2m-1)} v^{n+2m-1} \cdot \frac{(2m-2) \cdot (2n+2m)}{(2n+2m) \cdot (2n+2m+1)}$$

$$= -\frac{2^{n+2} \cdot ((m-1) \cdot m(m+1) \cdot \cdots \cdot (n+m)) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n+2m+1)}$$

& hence

$$= \pm r^{n+2}.$$

We see that, having the residues R^n, R^{n+1} the form that I have given to them, the residue R^{n+2} will have the same form. Therefore, it will only be a question of ascertaining the form of the first two residues R', R'', so as to establish what this first part of our proof had admitted as true in the form of hypothesis. And this is what will be the second part of the demonstration.

§. 12. Let us remember for this purpose that the first residue R' is the one that remains when dividing

$$\cos v = 1 - \frac{1}{2}v^2 + \frac{1}{2 \cdot 3 \cdot 4}v^4 \cdots \frac{1}{1 \cdots m}v^m \cdots \cdot \&c.$$

by

$$\sin v = v - \frac{1}{2 \cdot 3}v^3 + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}v^5 \cdots \frac{1}{1 \cdot \cdot (m+1)}v^{m+1} \cdots \cdot \&c.$$

Now, being the quotient that results from the division of the first term $= 1 : v$, we see that we shall have

$$R' = \cos v - \frac{1}{v} \cdot \sin v.$$

Therefore, multiplying the general term of the divisor,

$$\pm \frac{1}{1 \cdot 2 \cdot \cdots \cdot (m+1)} v^{m+1},$$

by $1 : v$, & subtracting the product

$$\pm \frac{1}{1 \cdot 2 \cdot \cdots \cdot (m+1)} \cdot v^m,$$

from the general term of the dividend

$$\pm \frac{1}{1 \cdot 2 \cdot \cdots \cdot m} \cdot v^m,$$

we shall have the general term of the first residue R'

$$r' = \pm \frac{m \cdot v^m}{1 \cdot \cdots \cdot (m+1)} \cdot$$

Now, $(m+1)$ being always an odd number, m will be an even number, & the first residue will be[24]

$$R' = -\frac{2}{2\cdot 3}v^2 + \frac{4}{2\cdot 3\cdot 4\cdot 5}v^4 - \frac{6}{2\cdot\cdots\cdot 7}v^6 + \&\text{c.}$$

just as we had assumed.

§. 13. The second residue R'' results from the division of

$$\sin v = v - \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 - \&\text{c.}\cdots \pm \frac{1}{1\cdot 2\cdots(m-1)}v^{m-1}$$

by the first residue that we just found[25]

$$R' = -\frac{2}{2\cdot 3}v^2 + \frac{4}{2\cdot 3\cdot 4\cdot 5}v^4 - \frac{6}{2\cdot\cdots\cdot 7}v^7 + \cdots \mp \frac{mv^m}{1\cdots(m+1)}$$

Now, being the quotient that results from the division of the first term $= -3 : v$, we see that we shall have

$$R'' = \sin v - \frac{3}{v}\cdot R'.$$

Multiplying therefore the general term of the divisor

$$\mp\frac{mv^m}{1\cdot\cdots\cdot(m+1)},$$

by $-3 : v$, & subtracting the product

$$\pm\frac{3mv^{m-1}}{1\cdot\cdots\cdot(m+1)},$$

from the general term of the dividend

[24] Note that in the series expansion for $\sin v$, $(m+1)$ takes the values $1, 3, 5\ldots$, and so m would take the values $0, 2, 4\ldots$ That makes the alternation in the signs that Lambert gives to r' to coincide with the one it actually has ($\mp$) because the first addend of R' is zero:

$$R' = +0 - \frac{2}{2\cdot 3}v^2 + \frac{4}{2\cdot 3\cdot 4\cdot 5}v^4 - \frac{6}{2\cdot\cdots\cdot 7}v^6 + \&\text{c.}$$

[25] There is an error in the third addend, as the expression should be written as follows:

$$R' = -\frac{2}{2\cdot 3}v^2 + \frac{4}{2\cdot 3\cdot 4\cdot 5}v^4 - \frac{6}{2\cdot\cdots\cdot 7}v^6 + \cdots \mp \frac{mv^m}{1\cdots(m+1)}$$

$$\pm \frac{1}{1 \cdot \cdots \cdot (m-1)} v^{m-1},$$

the general term of the second residue will be

$$\begin{aligned} r'' &= \pm \frac{v^{m-1}}{1 \cdot \cdots \cdot (m-1)} \mp \frac{3mv^{m-1}}{1 \cdot \cdots \cdot (m+1)} \\ &= \pm \frac{(m-2) \cdot m \cdot v^{m-1}}{1 \cdot \cdots \cdot (m+1)}. \end{aligned}$$

Substituting then for m the even numbers,[26] we shall have the second residue

$$R'' = -\frac{2 \cdot 4}{2 \cdot 3 \cdot 4 \cdot 5} v^3 + \frac{4 \cdot 6}{2 \cdots 7} v^5 - \frac{6 \cdot 8}{2 \cdots 9} v^7 + \&\text{c.}$$

again as we had assumed. Thus, having proved the form of the first two residues, it follows, by virtue of the first part of our proof, that the form of all the following residues is also proven.

§. 14. Now it is no longer necessary to prove separately the law of the progression of the quotients Q', Q'', Q''' &c. Because having demonstrated the law of the residues, it is by the same demonstration that any quotient will be (§. 10)

$$\pm Q^{n+2} = (2n+3) : v,$$

which, by virtue of the theory of continued fractions, gives

$$\tan v = \cfrac{1}{1 : v - \cfrac{1}{3 : v - \cfrac{1}{5 : v - \cfrac{1}{7 : v - \cfrac{1}{9 : v - \cfrac{1}{11 : v - 1 \ \&\text{c.}}}}}}}$$

from which we see in turn that *whenever the arc v is equal to an aliquot part of the radius, all these quotients will be increasing integer numbers in an arithmetic progression.*

And this is what we should note, since in the *Euclid*'s theorem cited above (§. 3) all quotients are assumed to be integer numbers. Thus, up to this point, the method prescribed by *Euclid* will be applicable to all these cases, in which the arc v is an

[26] Pay attention to the series expansion for R'.

aliquot part of the radius. But, even in these cases, there is another circumstance that should be stressed.[27]

§. 15. The problem proposed by *Euclid* is to *find the greatest common divisor of two integer numbers, which are not prime to each other*. This problem is solvable as long as one of the residues R', R'', R''' &c. . . . R^n becomes $= 0$, the preceding residue R^{n-1} not being equal to the unit, which following the 1st Prop. of the same book only occurs when the two proposed numbers are prime to each other, well understood that all quotients Q', Q'', Q''' &c. are assumed to be integer numbers. But we have just seen that this last supposition holds in the case at hand, provided $\frac{1}{v}$ is an integer. But, as for the residues R', R'', R''' &c. there is none that becomes $= 0$. On the contrary, considering the law of progression of the residues that we have just found, it is seen that they not only decrease without interruption, but also decrease faster than any geometric progression.[28] Although this continues to infinity, we can nevertheless apply the *Euclid*'s proposition to it. Because, by virtue of this proposition, *the greatest common divisor of* A, B *is at the same time the greatest common divisor of all residues* R, R', R'' &c. Now since these residues decrease in such a way that they ultimately become smaller than any assignable quantity, consequently *the greatest common divisor of* A, B, *is smaller than any assignable quantity*; which means that there is none, & therefore, A, B being two incommensurable quantities, *the*

$$\tan v = \frac{A}{B}$$

will be an irrational quantity every time the arc v is an aliquot part of the radius.

[27] In summary, this continued fraction will be within the conditions imposed by Euclid's algorithm as long as these quotients are integer numbers. In the next point, he applies this algorithm to these cases: since the residues never become zero, those divisions that aim to find the greatest common divisor never bring to an end. Furthermore, since the quotients are integers, we can conclude by applying the algorithm that the tangent will be an irrational number.

[28] This comparison with geometric progressions, which he will continue to emphasize, will be among Lambert's main supports to conclude the irrationality of his continued fraction (we will go into more detail about this later in this chapter).

§. 16. This is therefore, the limited use that can be made of the proposition of *Euclid*. It is now a question of extending it to all the cases in which the arc v is commensurable to the radius. For this purpose, & in order to prove still some other theorems,[29] I am going to take up once again the continued fraction

$$\tan v = \cfrac{1}{1:v - \cfrac{1}{3:v - \cfrac{1}{5:v - \cfrac{1}{7:v - 1\,\&\text{c.}}}}}$$

& making $1 : v = w$, I will transform it into

$$\tan v = \cfrac{1}{w - \cfrac{1}{3w - \cfrac{1}{5w - \cfrac{1}{7w - 1\,\&\text{c.}}}}}$$

§. 17. *Now, retaining the quotients* w, $3w$, $5w$ &c. *as much as we wish, we shall only have to do the reduction to obtain fractions that will express the tangent of v*

[29] Indeed, the next step is to extend the proof to all the values of the arc, but first Lambert demonstrates the convergence of this continued fraction (§. 17–§. 30), a rigurous approach that, at first glance, would fit better in the 19th century, a time when the worry about foundations were by far more intense. Nevertheless, it is known that foundational issues also worried to eighteenth-century mathematicians. Schubring in Schubring (2005, p. 285) draws attention to this:

> It is a widespread opinion to think of eighteenth-century mathematics as unconcerned with the foundations and as interested only in the further development of analysis [...] the mathematicians were, in contrast, very anxious to clarify basic concepts.

Topics such as infinitesimals, negative numbers or infinite series were widely discussed and studied, and it is likely that in this case Lambert was influenced by his context. Specifically (see Español et al. (2008)), there was a debate around the summability of divergent series in which Leibniz's series played an important role. The value $\frac{1}{2}$ for this series had been calculated analytically through the following formula:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

the more exactly the greater the number of quotients we have retained.[30] It is so e.g. that retaining 1, 2, 3, 4 &c. quotients, we find the fractions

(Footnote 29 continued)
by putting $x = 1$, but there were doubts as to whether the same series could be obtained through other finite expressions. Eventually, it was found that it could be obtained through the expression:

$$\frac{1+x}{1+x+x^2} = 1 - t^2 + t^3 - t^5 + t^6 - t^8 + \cdots$$

for $x = 1$, which would yield the value $\frac{2}{3}$. This would imply that:

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{2}{3}$$

Perhaps Lambert was aware of this and wanted to avoid criticism, ensuring that his infinite expression —now a continued fraction instead of a series— effectively came from a single expression, namely, $\frac{\sin v}{\cos v}$. Or maybe, more generally, his intention was to give his proof «all the geometric rigor» by retracing his steps (synthesis) to make it absolutely clear that the analytical development (analysis) was the correct one. (Serfati (1992, pp. 72–75) presents this part by means of such a dichotomy; it is interesting in this regard to consult (Mahoney 2000, pp. 739–740)). In any case, there is evidence that Lambert did indeed have knowledge about these disputes, and what is more interesting, about the case of continued fractions. In a letter sent to Euler on July 12, 1762, he writes (Bopp 1924, p. 28):

> For the same reason I did not take any part in a dispute, which the Academicians residing in Munich have considered appropriate to start with the Jesuit F[athers]. In this year's academic almanac, where I have given a popular idea of celestial movements, you will find a table of longitudes & latitudes of the Bavarian villages, and the calculation of finding approximately their distances. Since this calculation requires the extraction of the square root, I thought to please most readers by changing this extraction for two rules of three. The formula $\sqrt{a^2+b^2} = a + \cfrac{bb}{2a + \cfrac{bb}{2a + \cfrac{bb}{2a + \&c.}}}$ reveals all the mystery, which is not new. However, it is precisely about what the dispute resides, since the demonstration is demanded.

One would expect to find more references to continued fractions in this correspondence, given that Euler was one of the first to work on them systematically and that Lambert recognizes in Lambert (1766/1770, p. 162) that the motivation for the search for expressions in the form of a continuous fraction came precisely from Euler, and more specifically from his *Analysis infinitorum*, in which the expression as continued fraction for $\frac{e-1}{2}$ is included as an example Euler (1744, p. 325 Example III). However, the rest of the correspondence deals with physics. Be that as it may, it is still remarkable that Lambert devoted part of the proof to a convergence problem, something that others did not do (e.g., Euler in Euler (1744) or Legendre in Legendre I (1794), as mentioned by Baltus (2003, p. 10)).

[30] Proving that said continued fraction converges to tan v will consist of, given an arc v, showing that the sequence of fractions obtained by truncating this infinite expansion (the so-called convergents: $\frac{p_n}{q_n}$, $n \geq 1$):

$$\frac{1}{w}\ ,\quad \cfrac{1}{w - \cfrac{1}{3w}}\ ,\quad \cfrac{1}{w - \cfrac{1}{3w - \cfrac{1}{5w}}}\ ,\quad \ldots$$

$$\frac{1}{w},\quad \frac{3w}{3w^2-1},\quad \frac{15w^2-1}{15w^3-6w},\quad \frac{105w^3-10w}{105w^4-45w^2+1},$$

§. 18. But, *in order to make all these reductions in order, & to demonstrate at the same time the law of progression that these fractions obey*, we shall put first

$$\tan v = \frac{1}{w-a} = \cfrac{1}{w-\cfrac{1}{3w-a'}} = \cfrac{1}{w-\cfrac{1}{3w-\cfrac{1}{5w-a''}}} = \&c.$$

expressing by $a, a', a'', a'''....a^n, a^{n+1}, a^{n+2}....$ &c. the quantities that result from the quotients that we want to omit, so that in order to omit them we shall only have to put $a, a', a'', ...a^n$ &c. $= 0$.

§. 19. Now I say *that making* $a^{n+1} = 0$, *the fraction that results from the reduction of the quotients that we retain will have the form*[31]

$$\tan v = \frac{A - ma^n}{B - pa^n},$$

in which m, n, A, B *are not affected by* a^n.[32] Let us first suppose this form to be true, & it will be proved without difficulty that by retaining one more quotient, the fraction resulting from the reduction will have the same form. Since[33]

converges to $\tan v$. As an example, Lambert shows the expressions for the first four. On the other hand, his strategy will be to look for the law of recurrence for these fractions (§. 18–§. 22), whereupon he will find the general term dependent only on n. This will allow him to calculate the limit (§. 23–§. 28).

[31] Note that here, as will become clear later, A and B do not represent the sine and cosine, but serve as symbols. On the other hand, making an abuse of language, Lambert writes an equality, but it is not, since the term on the right represents the $(n+2)$-th convergent: if $a' = 0$, the formula expresses the 2-th convergent as a function of a; if $a'' = 0$, the formula expresses the 3-th convergent as a function of a'; in general, if $a^{n+1} = 0$, the formula expresses the $(n+2)$-th convergent as a function of a^n (the first convergent will be obtained by putting $a = 0$).

[32] As the reader will have noticed, the former group of four numbers should run as follows: m, p, A, B.

[33] See the note by A.S. in Appendix C. Here again there is a problem with indices: we just have to give some values to n in order to see that a^n does not match the expressions shown in the formula in section §. 18. In order to get a coincidence, the alluded expressions should be written like this:

$$\tan v = \frac{1}{w-a'} = \cfrac{1}{w-\cfrac{1}{3w-a''}} = \cfrac{1}{w-\cfrac{1}{3w-\cfrac{1}{5w-a'''}}} = \&c.$$

The only change introduced would be that:

$$a^n = \frac{1}{(2n+1)w - a^{n+1}} \ ,$$

we shall only have to substitute this value in the proposed form, & it will become[34]

$$\tan v = \frac{A(2n+1)w - m - A \cdot a^{n+1}}{B(2n+1)w - m - B \cdot a^{n+1}},$$

Since this form is the same,[35] it will suffice to show that it will be true for the member a', since then it will be true for all the following members a'', a''', $a^{IV} \ldots$, &c. Now, for member a' we have

$$\tan v = \cfrac{1}{w - \cfrac{1}{3w - a'}}$$

which by doing the reduction gives

$$\tan v = \frac{3w - a'}{3w^2 - 1 - wa'},$$

$$\text{If} \quad a'' = 0, \quad \text{said formula expresses the 2-th convergent as a function of} \quad a'$$

$$\text{If} \quad a''' = 0, \quad \text{said formula expresses the 3-th convergent as a function of} \quad a''$$

$$\ldots$$

$$\text{If} \quad a^{n+1} = 0, \quad \text{said formula expresses the (n+1)-th convergent as a function of} \quad a^n$$

(the first convergent is obtained by putting $a' = 0$).

[34] There is an error in the denominator. The expression should be written as follows:

$$\tan v = \frac{A(2n+1)w - m - A \cdot a^{n+1}}{B(2n+1)w - p - B \cdot a^{n+1}},$$

[35] Note that:

$$\frac{A(2n+1)w - m - A \cdot a^{n+1}}{B(2n+1)w - p - B \cdot a^{n+1}} = \frac{[A(2n+1)w - m] - A \cdot a^{n+1}}{[B(2n+1)w - p] - B \cdot a^{n+1}}$$

$$\equiv \frac{A - m \cdot a^{n+1}}{B - p \cdot a^{n+1}}$$

This expression would be the one corresponding to $a^{n+2} = 0$.

the form as we had supposed it.[36]

§. 20. Having then found[37]

$$\tan v = \frac{A - ma^n}{B - pa^n}$$

$$\tan v = \frac{A(2n+1)w - m - A \cdot a^{n+1}}{B(2n+1)w - m - B \cdot a^{n+1}},$$

we again substitute a^{n+1} for its value

$$a^{n+1} = \frac{1}{(2n+3)w - a^{n+2}},$$

& we shall have[38]

$$\tan v = \frac{\big[A(2n+1)w - m\big] \cdot (2n+3w) - A - \big[A(2n+1)w - m\big] \cdot a^{n+2}}{\big[B(2n+1)w - p\big] \cdot (2n+3w) - B - \big[B(2n+1)w - p\big] \cdot a^{n+2}}$$

§. 21. Therefore, by putting in each of these three values of $\tan v$, equal to zero the members a^n, a^{n+1}, a^{n+2}, we shall have the general form of the fractions that we try to find.[39]

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[36] Because:

$$\frac{3w - a'}{3w^2 - 1 - wa'} = \frac{3w - 1 \cdot a'}{(3w^2 - 1) - w \cdot a'} \equiv \frac{A - m \cdot a'}{B - p \cdot a'}$$

This expression would be the one corresponding to $a'' = 0$.

[37] The same error in the denominator of the second expression. It should be:

$$\tan v = \frac{A(2n+1)w - m - A \cdot a^{n+1}}{B(2n+1)w - p - B \cdot a^{n+1}}$$

[38] The expression should be written as follows:

$$\tan v = \frac{\big[A(2n+1)w - m\big] \cdot (2n+3)w - A - \big[A(2n+1)w - m\big] \cdot a^{n+2}}{\big[B(2n+1)w - p\big] \cdot (2n+3)w - B - \big[B(2n+1)w - p\big] \cdot a^{n+2}}$$

[39] They would correspond respectively to the $(n+1)$-th, $(n+2)$-th and $(n+3)$-th convergent (with the small change mentioned above, those fractions would represent respectively the n-th, $(n+1)$-th and $(n+2)$-th convergent).

$$\frac{A}{B},$$

$$\frac{A(2n+1)w-m}{B(2n+1)w-p},$$

$$\frac{\left[A(2n+1)w-m\right]\cdot(2n+3)w-A}{\left[B(2n+1)w-p\right]\cdot(2n+3)w-B}.$$

These three fractions resulting from the omission of a^n, a^{n+1}, a^{n+2} are consecutive, & we easily see *that the third is found by the two preceding ones, so that its numerator & its denominator can be calculated separately*. Since the numerator of the second fraction must be multiplied by the quotient corresponding to a^{n+1}, & from the product it is subtracted the numerator of the first fraction. The remainder will be the numerator of the third fraction. Its denominator is found in the same way by the denominators of the two preceding fractions.

§. 22. *To now obtain the fractions themselves,*[40] we shall only have to write the quotients in three columns with the numerators & the denominators of the first two fractions (§. 17), & the following numerators & denominators will be found by the simple operation that we have just indicated. Here is the model

Quotientes	numerators	denominators
	1	w
$5w$	$3w$	$3w^2-1$
$7w$	$15w^2-1$	$15w^3-6w$
$9w$	$105w^3-10w$ . . .	$105w^4-45w^2+1$
$11w$	$945w^4-105w^2+1$	$945w^5-420w^3+15w$
& c.	$10395w^5-1260w^3+21w$	$10395w^6-4725w^4+210w^2-1$
	& c.	& c.

Which gives the fractions

$$\frac{1}{w},\quad \frac{3w}{3w^2-1},\quad \frac{15w^2-1}{15w^3-6w},\quad \frac{105w^3-10w}{105w^4-45w^2+1}\quad \&\text{c.}$$

each of which expresses more exactly the tangent of v than those that precede it.[41]

§. 23. Now, despite the fact that by means of the rule that we have just given (§. 21), each one of these fractions can be found by the two that immediately precede it, *it will be convenient, to avoid here again a kind of induction, to provide & prove the general expression*. Let us begin first of all by noting that the coefficients of each

[40] He refers to the non-recurring general expression for the succession, that is to say, that expression dependent only on the position of the fraction in the sequence (n). This is what he will search for in §. 22–§. 28.

[41] This is precisely what he will prove.

vertical column follow a very simple law in which their factors are in part figurative numbers & in part odd numbers. Here they are determined

Fraction	Quotient	Denominator
1ª		w
2ª	$5w$	$3 \cdot w^2 - 1 \cdot 1$
3ª	$7w$	$3 \cdot 5 \cdot w^3 - 2 \cdot 3w$
4ª	$9w$	$3 \cdot 5 \cdot 7w^4 - 3 \cdot 3 \cdot 5w^2 + 1 \cdot 1$
5ª	$11w$	$3 \cdot \cdot 9w^5 - 4 \cdot 3 \cdot 5 \cdot 7w^3 + 3 \cdot 5w$
6ª	$13w$	$3 \cdots 11w^6 - 5 \cdot 3 \cdot \cdot 9w^4 + 6 \cdot 5 \cdot 7w^2 - 1 \cdot 1$
7ª	$15w$	$3 \cdots \cdot 13w^7 - 6 \cdot 3 \cdots 11w^5 + 10 \cdot 5 \cdot 7 \cdot 9w^3 - 4 \cdot 7w$
& c.	& c.	&c.
Fraction	Quotient	Numerator
1ª		1
2ª	$5w$	$3w$
3ª	$7w$	$3 \cdot 5w^2 - 1 \cdot 1$
4ª	$9w$	$3 \cdot 5 \cdot 7w^3 - 2 \cdot 5w$
5ª	$11w$	$3 \cdot \cdot 9w^4 - 3 \cdot 5 \cdot 7w^2 + 1 \cdot 1$
6ª	$13w$	$3 \cdots 11w^5 - 4 \cdot 5 \cdot 7 \cdot 9w^3 + 3 \cdot 7w$
7ª	$15w$	$3 \cdots \cdot 13w^7 - 5 \cdot 5 \cdot \cdot 11w^4 + 6 \cdot 7 \cdot 9w^2 - 1 \cdot 1$
& c.	& c.	&c.

§. 24. *This observation gives us the means to find the general expression*[42] *for any of these fractions*. Consider the n-th of these fractions, & we shall have its

[42] If we look at the expressions for the denominators (similar reasoning for the numerators), we see that:

1. The first factor of all of them is a product of odd numbers starting with 1, followed by a power of w:

$$1 \cdot w, \quad 1 \cdot 3 \cdot w^2, \quad 1 \cdot 3 \cdot 5 \cdot w^3, \quad 1 \cdot 3 \cdot 5 \cdot 7 \cdot w^4, \quad \ldots \quad [1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k-1)] \cdot w^k$$

   the general expression of which is ($k = n$):

$$[1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-1)] \cdot w^n$$

2. In the second factors what we have is the sequence of natural numbers (with general term: $k,\ k \geq 1$), followed by products of odd numbers starting with 1, and a power of w:

$$2 \cdot 1 \cdot 3 \cdot w, \quad 3 \cdot 1 \cdot 3 \cdot 5 \cdot w^2, \quad 4 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot w^3, \quad \ldots$$

   giving rise to:

$$[(k+1) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k+1)] \cdot w^k = [2(k+1) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2k+1)] \cdot \frac{w^k}{2}$$

   and therefore ($k = n-2$):

$$[(2n-2) \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n-3)] \cdot \frac{w^{n-2}}{2}$$

3. In the third factors it appears the sequence (1 does not count multiply) 3, 6, 10, ... (with general term $\frac{(k+2)(k+1)}{2}$, $k \geq 1$), followed by products of odd numbers starting with 1 (omiting 3), and a power of w:

$$3 \cdot 1 \cdot 5 \cdot w, \quad 6 \cdot 1 \cdot 5 \cdot 7 \cdot w^2, \quad 10 \cdot 1 \cdot 5 \cdot 7 \cdot 9 \cdot w^3, \quad \ldots$$

Denominator

$$= w^n\,[1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)] - \frac{w^{n-2}}{2}\cdot[(2n-2)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-3)]$$

$$+ \frac{w^{n-4}}{2\cdot 3\cdot 4}\cdot[(2n-4)\cdot(2n-6)\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-5)]$$

$$- \frac{w^{n-6}}{2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot[(2n-6)\cdot(2n-8)\cdot(2n-10)\cdot 1\cdot 3\cdot 5\cdots(2n-7)]$$

$$+ \frac{w^{n-8}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}\cdot[(2n-8)\cdot(2n-10)(2n-12)(2n-14)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-9)]$$

$$- \&c.$$

Numerator

$$= w^{n-1}\cdot[1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-1)] - \frac{w^{n-3}}{2\cdot 3}\cdot[(2n-4)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-3)]$$

$$+ \frac{w^{n-5}}{2\cdot 3\cdot 4\cdot 5}\cdot[(2n-6)(2n-8)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-5)]$$

$$- \frac{w^{n-7}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\,[(2n-8)\cdot(2n-10)(2n-12)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-7)]$$

$$+ \frac{w^{n-9}}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}\cdot[(2n-10)\cdot(2n-12)\cdot(2n-14)\cdot(2n-16)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2n-9)]$$

$$- \&c.$$

The issue therefore is nothing but to demonstrate the universality.

§. 25. This is what will be achieved by admitting this form for the n-th fraction, from where, substituting[43] $(n-1),\ (n-2)$ in the place of n, those for the $(n-1)$-th

which gives rise to:

$$[(k+2)\cdot(k+1)\cdot 1\cdot 5\cdot 7\cdot 9\cdot\cdots\cdot(2k+3)]\cdot\frac{w^k}{2} = [(2k+4)\cdot(2k+2)\cdot 1\cdot 3\cdot 5\cdot 7\cdot\cdots\cdot(2k+3)]\cdot\frac{w^k}{2\cdot 3\cdot 4}$$

and therefore ($k = n-4$):

$$[(2n-4)\cdot(2n-6)\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-5)]\cdot\frac{w^{n-4}}{2\cdot 3\cdot 4}$$

All this is sufficient to postulate a general term.

[43] Just in case Lambert's language is not sufficiently clear: by using the expression of the n-th denominator, he obtains both the $(n-1)$-th and the $(n-2)$-th denominators by substituting in said expression n for $n-1$ and $n-2$. So, having the three expressions, he applies the rule given in

& $(n-2)$-th are deduced. Then we proceed according to the rule §. 21 by deducing both the denominator and the numerator of the n-th fraction from the two preceding ones as we have just found them by the first operation. And with that, the form of the n-th fraction must be reproduced, as we have just given it. It is clear that this process leads to establishing that if two consecutive fractions have this form, the one that follows them will also have it, & that, therefore, the fractions of the preceding table, which are the first, having this form, it will follow that all the following will also have it.

§. 26. If therefore, in order to abbreviate this proof, we want to adhere to the general term, it will be nonetheless necessary to calculate the numerator & the denominator separately, for no other reason than to simplify the calculation. Because moreover, both will be calculated following the same rule (§. 21). Let us start with the *denominator*, & taking the m-th term of its general expression for the n-th fraction, we shall also have to take the m-th term for the $(n-1)$-th fraction, but we shall only take the $(m-1)$-th term for the $(n-2)$-th fraction.[44] We see that it is necessary to act in this way with respect to the dimensions or exponents of the letter w.

§. 27, Now the m-th term of the n-th fraction for the denominator is[45]

$$M = \frac{w^{n-2m+2} \cdot [(2n-2m+2)\cdot(2n-2m)\cdot(2n-2m-2)\cdot\cdots\cdot(2n-4m+6)]\cdot[1\cdot 3\cdot 5\cdots(2n-2m+1)]}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(2m-2)}$$

from where, by substituting $(n-1)$ in the place of n, the m-th term of the $(n-1)$-th fraction we find

$$M' = \frac{w^{n-2m+1} \cdot [(2n-2m)\cdot(2n-2m-2)\cdot\cdots\cdot(2n-4m+4)]\cdot[1\cdot 3\cdot 5\cdot\cdots\cdot(2n-2m-1)]}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot\cdots\cdot(2m-2)}$$

And by substituting $(n-2)$ in the place of n, & $(m-1)$ in the place of m, the $(m-1)$-th term of the $(n-2)$-th fraction is found

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§. 21 in order to verify that both members coincide, that is:

$$q_n = (2n-1)w \cdot q_{n-1} - q_{n-2}$$

[44] Lambert will prove, term by term, that the general expressions satisfy the relation given in §. 21, that is:

$$q_n^{(m)} = (2n-1)w \cdot q_{n-1}^{(m)} - q_{n-2}^{(m-1)}$$

for the denominators (Lambert uses, of course, another notation, namely: M, M', M'') and:

$$p_n^{(m)} = (2n-1)w \cdot p_{n-1}^{(m)} - p_{n-2}^{(m-1)}$$

for the numerators (in this case he uses N, N', N''). Paying attention to the denominators (the same with the numerators) of the table at point §. 22, it is clear that the number of terms to be taken in the $(n-2)$-th fraction is one less than in the remaining two, something that, as Lambert mentions bellow, is related to the power of w.

[45] It is an easy exercise to arrive at this general term by looking at the expressions of §. 24.

$$M'' = \frac{w^{n-2m+2} \cdot [(2n-2m) \cdot (2n-2m-2) \cdot \cdots \cdot (2n-4m+6)] \cdot [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-2m-1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (2m-4)}$$

Now by the rule §. 21 it must be

$$M = (2n-1)w \cdot M' - M''$$

which makes it possible for us to free these three expressions from all the factors that are common to them, by putting them $= P$. Therefore we shall have[46]

$$+M = \frac{P \cdot w \cdot (2n-2m+2) \cdot (2n-2m+1)}{(2m-2) \cdot (2m-3)}$$

$$+M' = \frac{P \cdot (2n-4m+4)}{(2m-2) \cdot (2m-3)}$$

$$-M'' = P \cdot w.$$

Or putting

$$\frac{P}{(2m-2) \cdot (2m-3)} = Q,$$

we shall have[47]

$$\begin{aligned} +M &= Qw \cdot (2n-2m+2) \cdot (2n-2m+1) \\ +M' &= Q \cdot (2n-4m+2) \\ -M'' &= Qw \cdot (2m-2) \cdot (2m-3). \end{aligned}$$

From that, by multiplying, we shall have[48]

$$\begin{aligned} (2n-1)wM' &= Qw \cdot (4n^2 - 8mn + 6n + 4m - 4) \\ -M'' &= Qw \cdot (4m^2 - 10m + 6) : \end{aligned}$$

from where

[46] Using the factorial notation, that common factor would be:

$$P = \frac{w^{n-2m+1}(2n-2m)(2n-2m-2)\cdots(2n-4m+6) \cdot (2n-2m-1)!}{(2m-4)!}$$

The introduction of P, and in the next step, of Q, has no other objective than to simplify the calculations (the same for the numerator).

[47] There is a typo in the second expression. It should be written as follows:

$$+M' = Q \cdot (2n-4m+4)$$

[48] The colon in the second expression is typographical; it has nothing to do with division.

$$(2n-1)wM' - M'' = Qw(4n^2 - 8nm + 6n + 4m^2 - 6m + 2).$$

But also

$$M = Qw \cdot (2n-2m+2)(2n-2m+1) = Qw(4n^2 - 8nm + 6n + 4m^2 - 6m + 2).$$

Therefore, these two values being equal, we see that it is

$$M = (2n-1)w \cdot M' - M'',$$

& that consequently the form which we have given to the general term is such as it should be.

§. 28. Let us now turn to the *numerator*. The m-th term of the numerator of the n-th fraction must be

$$+N = \frac{w^{n-2m+1} \cdot [(2n-2m) \cdot (2n-2m-2) \cdot \cdots \cdot (2n-4m+4)] \cdot [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-2m+1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (2m-1)}$$

from where, by substituting $(n-1)$ in the place of n, we shall have the same m-th term for the $(n-1)$-th fraction,

$$+N' = \frac{w^{n-2m} \cdot [(2n-2m-2) \cdot (2n-2m-4) \cdot \cdots \cdot (2n-4m+2)] \cdot [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-2m-1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (2m-1)}$$

And by substituting $(n-2)$, $(m-1)$, in the place of n, m, we shall have the $(m-1)$-th term of the $(n-2)$-th fraction,

$$-N'' = \frac{w^{n-2m+1} \cdot [(2n-2m-2) \cdot (2n-2m-4) \cdot \cdots \cdot (2n-4m+4)] \cdot [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-2m-1)]}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (2m-3)}$$

Therefore, putting the common factors to these three expressions $= P$, we shall have[49]

$$+N = \frac{Pw \cdot (2n-2m) \cdot (2n-2m+1)}{(2m-1) \cdot (2m-2)}$$

$$+N' = \frac{P \cdot (2n-4m+2)}{(2m-1) \cdot (2m-2)}$$

$$-N'' = Pw,$$

or doing $P = Q \cdot (2m-1) \cdot (2m-2)$, we shall have

[49] Now P would be:

$$P = \frac{w^{n-2m}(2n-2m-2)(2n-2m-4)\cdots(2n-4m+4) \cdot (2n-2m-1)!}{(2m-3)!}$$

$$+N = Qw \cdot (2n - 2m) \cdot (2n - 2m + 1)$$
$$+N' = Q \cdot (2n - 4m + 2)$$
$$-N'' = Qw \cdot (2m - 1) \cdot (2m - 2).$$

But it must be

$$N = (2n - 1)w \cdot N' - N'',$$

then, by substituting the values found, we shall have

$$(2n - 1)wN' = Qw \cdot (4nn - 8nm + 2n + 4m - 2)$$
$$-N'' = Qw(4m^2 - 6m + 2),$$

from where

$$(2n - 1)wN' - N'' = Qw(4n^2 - 8nm + 2n + 4m^2 - 2m).$$

But the same value results from

$$N = (2n - m) \cdot (2n - 2m + 1) \cdot Qw.$$

It then follows from this that the form of the general term is such as it should be.

§. 29. Let us take hence the general expressions that we have given in §. 24 & let us divide the denominator by its first term,[50] & we shall have the series[51]

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[50] Having then established the general terms for the numerator and denominator, Lambert's strategy is as follows:

$$\frac{p_n}{q_n} = \frac{p_n/q_n^{(1)}}{q_n/q_n^{(1)}} \xrightarrow[n\to\infty]{} \tan v$$

since (§. 29):

$$\frac{q_n}{q_n^{(1)}} \xrightarrow[n\to\infty]{} \cos v$$

and because (§. 30):

$$\frac{p_n}{q_n^{(1)}} \xrightarrow[n\to\infty]{} \sin v$$

[51] A parenthesis is missing in the third addend:

$$1 - \frac{w^{-2}}{2} \cdot \frac{2n-2}{2n-1} + \frac{w^{-4}}{2 \cdot 3 \cdot 4} \cdot \frac{(2n-4) \cdot (2n-6)}{(2n-1) \cdot (2n-3)} - \frac{w^{-6}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{(2n-6) \cdot (2n-8)(2n-10)}{(2n-1)(2n-3)(2n-5)}$$
$$+ \frac{w^{-8}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} \cdot \frac{(2n-8) \cdot (2n-10) \cdot (2n-12) \cdot (2n-14)}{(2n-1) \cdot (2n-3) \cdot (2n-5) \cdot (2n-7)} - \text{\&c.}$$

$$1-\frac{w^{-2}}{2}\cdot\frac{2n-2}{2n-1}+\frac{w^{-4}}{2\cdot3\cdot4}\cdot\frac{(2n-4)\cdot(2n-6)}{(2n-1)\cdot(2n-3)}-\frac{w^{-6}}{2\cdot3\cdot4\cdot5\cdot6}\cdot\frac{(2n-6)\cdot(2n-8)(2n-10)}{(2n-1)(2n-3)(2n-5)}$$
$$+\frac{w^{-8}}{2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}\cdot\frac{(2n-8)\cdot(2n-10)\cdot(2n-12)\cdot(2n-14)}{(2n-1)\cdot(2n-3)\cdot(2n-5)(2n-7)}-\&c.$$

which, by substituting $v=w^{-1}$, & putting $n=\infty$, gives[52]

$$1-\frac{v^2}{2}+\frac{v^4}{2\cdot3\cdot4}-\frac{v^6}{2\cdot3\cdot4\cdot5\cdot6}+\&c.$$

which is the cosine of v, & therefore the denominator that we have used (§. 5) in order to find the quotients $w,\ 3w$ &c.

§. 30. Let us now divide the general expression of the numerator (§. 24) by the same first term of the denominator, & we shall have the series

$$w^{-1}-\frac{w^{-3}}{2\cdot3}\cdot\frac{2n-4}{2n-1}+\frac{w^{-5}}{2\cdot3\cdot4\cdot5}\cdot\frac{(2n-6)\cdot(2n-8)}{(2n-1)\cdot(2n-3)}$$
$$-\frac{w^{-7}}{2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot\frac{(2n-8)\cdot(2n-10)\cdot(2n-12)}{(2n-1)\cdot(2n-3)\cdot(2n-5)}$$
$$+\&c.$$

Which now for $n=\infty$, gives the series

$$v-\frac{1}{2\cdot3}v^3+\frac{1}{2\cdot3\cdot4\cdot5}v^5-\&c.$$

which is $=\sin v$, & therefore the numerator which we have used[53] §. 5.

§. 31. We further see in this way *that, no matter how large the first term of the two general formulas may be* (§. 24), *the second term, & even more the following ones, will not only be smaller, but smaller than the* $\frac{1}{2},\ \frac{1}{2\cdot3},\ \frac{1}{2\cdot3\cdot4}$ *&c. part of the first term.*[54] But by substituting n successively for $1,\ 2,\ 3,\ 4$ &c. to infinity, *the first term* being the product of the odd numbers $1\cdot3\cdot5\cdot7$ &c. *will grow faster than any increasing geometric progression;*[55] we also see *that even if the term* $2,\ 4,\ 6$ *&c. is subtracted, this does not impede the sum of the terms from growing faster than any*

[52] This is because the coefficients that multiply the successive powers of w tend to 1 when $n\to\infty$. Note that the concept of uniform convergence had not yet made its appearance —it would take about a century— so this justification is quite rigorous by the canons of its time.

[53] This concludes the second part of the proof. However, before entering the last part —the one referring to irrationality— and after certain considerations (§. 31), Lambert looks for the series expassion for the tangent from its expression in continued fraction (§. 32–§. 37).

[54] From the first and second formula, alternately.

[55] Note that, if $a\neq0$:

increasing geometric progression.[56] And I make this observation here, because I will make use of it in the rest of this Memoir. Here is what comes first.

§. 32. The aim is *to determine the law according to which the fractions*

$$\frac{1}{w}, \quad \frac{3w}{3w^2-1}, \quad \frac{15w^2-1}{15w^3-6w}, \quad \text{\&c.}$$

approximate the value of the tangent.[57] For this effect, we shall only have to subtract each one from the one that follows it, & the residues will be

$$\frac{1}{w\cdot(3w^2-1)}, \quad \frac{1}{(3w^2-1)\cdot(15w^3-6w)}, \quad \text{\&c.}$$

These residues show how much larger each of the fractions is than the one that precedes it. But let us show in general that *all the numerators are* $= 1$, & *that all the denominators are the product of those of the two fractions the difference of which is indicated by these residues.*

§. 33. To this effect, let us take up again the three general formulas that we have given in §. 21 & which are

$$\frac{A}{B},$$

$$\frac{A(2n+1)w-m}{B(2n+1)w-p},$$

$$\frac{\big[A(2n+1)w-m\big](2n+3)w-A}{\big[B(2n+1)w-p\big](2n+3)w-B}.$$

Now, subtracting the first from the second, the residue will be

$$=\frac{Ap-Bm}{B\cdot[B(2n+1)w-p]}.$$

But the numerator of this residue is the same as that resulting from the subtraction

$$\frac{A}{B}-\frac{m}{p}=\frac{Ap-Bm}{B\cdot p}.$$

$$\frac{n!}{a^n}\xrightarrow[n\to\infty]{}\infty$$

(by using Stirling, for example).

[56] Precisely because of what has been observed in the first paragraph of this very point.

[57] The sentence ends here with a question mark that has been decided to omit for a better understanding of the text.

Now $\frac{m}{p}$ being the fraction that precedes the fraction $\frac{A}{B}$, we see that the numerator of all these residues is the same, & that the denominator is the product of those of the fractions the difference of which is indicated by these residues. Therefore, starting with any one of the fractions $\frac{m}{p}$, the residues will be

$$\frac{1}{p \cdot B}, \quad \frac{1}{B\,[B(2n+1)w - p]} \quad \&\text{c.}$$

§. 34. Let us now observe, *that all these residues being added to the first fraction which is set as a basis, the sum will always express the tangent of* v, *so that in general it will be*[58]

$$\tan v = \frac{m}{p} + \frac{1}{p \cdot B} + \frac{1}{B \cdot [B(2n+1)w - p]} + \&\text{c.}$$

& hence

$$\tan v = \frac{1}{w} + \frac{1}{w(3w^2-1)} + \frac{1}{(3w^2-1)\cdot(15w^3-6w)} + \&\text{c.}$$

$$\tan v = \frac{3w}{3w^2-1} + \frac{1}{(3w^2-1)(15w^3-6w)} + \&\text{c.}$$

$$\tan v = \frac{15w^2-1}{15w^3-6w} + \frac{1}{(15w^3-6w)\cdot(105w^4-45w^2+1)} + \&\text{c.}$$

&c.

We see therefore from what we have said (§. 31.) *that all these series are more convergent than any decreasing geometric progression*. Let e.g. $v = w = 1$, & the tangent of this arc will be $= 1,55740772\ \cdots$.[59]

[58] Actually, this responds to a more general framework. Given a continued fraction:

$$f = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

it can be seen as:

$$f = \frac{p_1}{q_1} + \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right) + \left(\frac{p_3}{q_3} - \frac{p_2}{q_2}\right) + \cdots$$

[59] See the note by A.S. in Appendix C.

$$= 1 + \frac{1}{1 \cdot 2} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540} + \frac{1}{540 \cdot 5879} +$$

$$\frac{1}{5879 \cdot 75587} + \frac{1}{75587 \cdot 1147426} + \&\text{c.}$$

And for every arc $v < 1$, we shall have an even more convergent series.

§. 35. Let us put now $w = \omega : \varphi,\ v = \varphi : \omega$, so that φ, ω are any integer numbers prime to each other. *We shall only have to substitute these values, & we shall have*

$$\tan\left(\frac{\varphi}{\omega}\right) = \cfrac{\varphi}{\omega - \cfrac{\varphi\varphi}{3\omega - \cfrac{\varphi\varphi}{5\omega - \cfrac{\varphi\varphi}{7\omega - \cfrac{\varphi\varphi}{9\omega - \ \&\text{c.}}}}}}$$

§. 36. Then *the fractions that approximate*[60] *the value of the* tan $\frac{\varphi}{\omega}$ *will be*

$$\frac{\varphi}{\omega}, \quad \frac{3\omega\varphi}{3\omega^2 - \varphi^2}, \quad \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\varphi^2\omega}, \quad \frac{105\omega^3\varphi - 10\omega\varphi^3}{105\omega^4 - 45\omega^2\varphi^2 + \varphi^4}, \quad \&\text{c.}$$

in such a way that any two of these consecutive fractions being

$$\frac{m}{p},$$

$$\frac{A}{B},$$

the one that follows will be

$$\frac{A(2n+1)\omega - m\varphi^2}{B(2n+1)\omega - p\varphi^2}.$$

§. 37. Finally, *the differences of these fractions will be*

$$\frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)}, \quad \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)}, \quad \&\text{c.}$$

& the

60 Lambert writes: *fractions approchantes*.

$$\tan\frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2-\varphi^2)} + \frac{\varphi^5}{(3\omega^2-\varphi^2)\cdot(15\omega^3-6\omega\varphi^2)} + \quad \&c.$$

Now, I say that this tangent will never be commensurable to the radius whatever the integer numbers ω, φ may be.

§. 38. In order to prove this theorem,[61] let us put

$$\tan\frac{\varphi}{\omega} = \frac{M}{P},$$

in such a way that M, P are quantities expressed in any way, even, if we want, by decimal sequences, which can always be done even if M, P, are integer numbers, since we shall only have to multiply one & another by some irrational quantity.[62] We can even suppose, if we wish, $M = \sin\frac{\varphi}{\omega}$, $P = \cos\frac{\varphi}{\omega}$, as we have done above (§. 5.). And it is clear that, even if the $\tan\frac{\varphi}{\omega}$ were rational, the same would not always have to happen with the $\sin\frac{\varphi}{\omega}$ & the $\cos\frac{\varphi}{\omega}$.

§. 39. Since the fraction

$$\frac{M}{P}$$

[61] From this point to §. 51 the text is devoted to the demonstration of irrationality, and for this purpose, Lambert draws on the *reductio ad absurdum* method taking a rational arc $v \in \mathbb{Q}$ and assuming that its tangent is also rational $\tan v \in \mathbb{Q}$. Based on this assumption, he builds a succession of real numbers (using our language) $R', R'', R''', \ldots, R^n, \ldots$, such that $R^n = D \cdot r^n$ —r^n being non zero integer numbers and $D \notin \mathbb{Q}$ being a constant—, and such that $R^n \xrightarrow[n\to\infty]{} 0$. This necessarily entails that $D = 0$, which contradicts the fact that $D \notin \mathbb{Q}$.

[62] This phrase means that we can always suppose that the tangent (and whatever quantity, actually) is the quotient of two quantities M and N, and regardless of the nature of both —«even if M, P, are integer numbers»—, said quotient can always be transformed into a fraction in which the numerator and denominator «are quantities expressed in any way, even if we want, by decimal sequences [*suites décimales*]» by multiplying both by an irrational quantity. However, if we take a closer look at the phrase, it seems to indicate that Lambert thought of irrationals as quantities made up of an actual amount of decimal digits, because if those integers were multiplied by decimal quantities with a finite number of digits, we would obtain quantities with a finite number of digits that would also be immediately transformed into a rational fraction, returning to the case that numerator and denominator are integers. In order to not lose generality and avoid this situation, it would be necessary to choose an infinite (non-periodic) sequence of decimal digits —irrationals—, which is in fact what he does. We have evidence that supports this interpretation in the already mentioned Part V of Lambert (1766/1770). In this work, Lambert associates each convergent of the continued fraction with a decimal approximation for π, which he calls «Ludolphian numbers» (he also refers to them right at the beginning of this *Mémoire*) —not π—, which appears to show an awareness of π as formed by an actual amount of decimal digits, the result of not truncating the continued fraction in any step, something in itself notorious, and representative of the turning point (1600–1800) from considering decimals as mere approximative tools in the study of certain magnitudes (900–1600) —with S. Stevin as the later representative—, to think of them as the object of study in itself (1750–1950) Ferreirós (2015, pp. 146–149).

exactly expresses the tangent of $\frac{\varphi}{\omega}$, it must give all the quotients w, $3w$, $5w$ &c. which in the present case are

$$+\frac{\omega}{\varphi}, \quad -\frac{3\omega}{\varphi}, \quad +\frac{5\omega}{\varphi}, \quad -\frac{7\omega}{\varphi}, \quad + \quad \&c.$$

§. 40. Hence, if the tan $\frac{\varphi}{\omega}$ is rational, it is clear that M will be to P as an integer number μ to an integer number[63] π, in such a way that, if μ, π, are primes to one another, we shall have

$$M : \mu = P : \pi = D,$$

& D will be the greatest common divisor of M, P. And as it is reciprocally

$$M : D = \mu,$$
$$P : D = \pi,$$

we see that M, P being supposed irrational quantities, their greatest common divisor will also be an irrational quantity, which is the smaller, the larger the quotients[64] μ, π are.

§. 41. Here are therefore *the two suppositions whose incompatibility will have to be shown*. Let us divide first[65] P by M, & the quotient must be $= \omega : \varphi$. But since $\omega : \varphi$ is a fractional number, let us divide φP by M, & the quotient ω will be φ-tuple of $\omega : \varphi$. It is clear that we can divide it by φ, whenever we want to do so.

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63 It is clear that Lambert is not using here the Greek word π to represent the ratio between the circumference and its diameter, although this symbol had already been used before for this purpose (for the first time) by William Jones in 1706. It is not surprising anyway, that an author of the time did not use a notation that was just beginning to take form (note that information did not flow as it does today, an important detail that we must not ignore), although it is still curious that out of all the possible symbols he had chosen precisely that one. Having said that, Lambert in Lambert (1766/1770) does use the Greek word π to represent this ratio (for instance in p. 147).

64 In summary, what we have is that the tangent being able to be expressed as:

$$\tan v = \frac{\sin v}{\cos v} = \frac{M}{P},$$

with $M \notin \mathbb{Q}$ and $P \notin \mathbb{Q}$, and moreover being supposed to be rational $\tan v \in \mathbb{Q}$, there will exist two numbes μ and π such that:

$$\tan v = \frac{M}{P} = \frac{\mu}{\pi},$$

and therefore there will also exist $D \notin \mathbb{Q}$, such that $\mu \cdot D = M$ and $\pi \cdot D = P$, and so:

$$\frac{M}{D} = \mu \in \mathbb{Z} \quad \text{and} \quad \frac{P}{D} = \pi \in \mathbb{Z} \tag{5.1}$$

65 Lambert begins, step by step (§. 41, 42), the construction of the succession alluded to before, which will end with a proposal for a general term (§. 42, 43) and its proof (§. 44). Since the steps taken by Lambert are not entirely clear in the text, they will be followed in parallel in the footnotes.

Here it will not be necessary, since it is enough for us that it is an integer. Having therefore obtained the quotient ω by dividing φP by M, let the residue be $= R'$. This residue will equally be φ-tuple of what it would have been, & which we shall take into account. Now, since $P : D = \pi$ is an integer number, $\varphi P : D = \varphi\pi$ will also be an integer number. Finally $R' : D$ will also be an integer number. Since, because

$$\varphi P = \omega M + R',$$

we shall have

$$\frac{\varphi P}{D} = \frac{\omega M}{D} + \frac{R'}{D}.$$

But

$$\varphi P : D = \varphi\pi,$$
$$\omega M : D = \omega\mu,$$

hence

$$\varphi\pi = \omega\mu + \frac{R'}{D},$$

which gives

$$\frac{R'}{D} = \varphi\pi - \omega\mu = \text{integer number},$$

which we shall put $= r'$, so that

$$\frac{R'}{D} = r'.$$

Therefore the residue of the first division will still have D as divisor, which is the greatest common divisor of[66] M, P.

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[66] **First step:**

If $\tan\frac{\varphi}{\omega} = \frac{M}{P}$,

$$\tan\frac{\varphi}{\omega} = \frac{M}{P} = \frac{1}{\frac{P}{M}} = \frac{1}{\frac{\omega}{\varphi} + \frac{{}^1R}{M}},$$

where ${}^1R$ is the residue of the first division (the one that Lambert says that must be taken into account). Let us note that the reason why in the previous expression —as well as those that will come in the following steps— appears $+$ where there should be $-$, is that the sign is contained in ${}^1R$, and therefore in the R' that will appear below. At the end of this process it will be seen how this sign is taken into account. Having said that, now, multiplying by φ in the adequate place:

$$\frac{1}{\varphi\cdot\frac{P}{M}} = \frac{1}{\omega + \frac{{}^1R\cdot\varphi}{M}} = \frac{1}{\omega + \frac{R'}{M}},$$

§. 42. Let us pass now to the second division. The residue R' being φ-tuple of what it would be if we had divided P, instead of φP, by M, it will have to be taken into account in this second division, by dividing φM, instead of M, by R', in order to obtain the second quotient, which is $= 3\omega : \varphi$. But, in order to avoid the fractional quotient here again, let us divide $\varphi^2 M$ by R', to obtain the quotient 3ω, integer number. Let the residue be $= R''$, & it will be

$$\varphi^2 M = 3\omega R' + R'',$$

hence dividing by D,

$$\frac{\varphi^2 M}{D} = \frac{3\omega R'}{D} + \frac{R''}{D}.$$

But

$$\frac{\varphi^2 M}{D} = \varphi^2 m = \text{integer number},$$
$$\frac{3\omega R'}{D} = 3\omega r' = \text{integer number},$$

hence

$$\varphi^2 m = 3\omega r' + \frac{R''}{D},$$

which gives

$$\frac{R''}{D} = \varphi^2 m - 3\omega r' = \text{integer number},$$

which we shall put $= r''$, so that

$$\frac{R''}{D} = r''.$$

we obtain that $\varphi P = \omega M + R'$, and therefore:

$$\begin{cases} R' = \varphi P - \omega M \\ \\ \dfrac{R'}{D} = r' \in \mathbb{Z} \end{cases}$$

The second equality (the integer character of $\frac{R'}{D}$), is obtained directly from dividing the first equality by D and taking into account (5.1).

Therefore the greatest common divisor of $M,\ P,\ R'$, is still of the second residue[67] R''.

§. 43. Let the following residues be $\cdots R''', R^{IV} \cdots\cdots R^n, R^{n+1}, R^{n+2} \cdots$, which correspond to the φ-tuples quotients $\cdots 5\omega, 7\omega \cdots\cdot (2n-1)\omega, (2n+1)\omega, (2n+3)\omega \cdots$, & the aim is to prove in general that if any two arbitrary residues $R^n,\ R^{n+1}$, which follow each other immediately, still have D as divisor, the next residue R^{n+2} will have it too, so that if, by doing

$$R^n : D = r^n,$$
$$R^{n+1} : D = r^{n+1},$$

$r^n,\ r^{n+1}$ are integer numbes, we shall also have[68]

$$R^{n+2} : D = r^{n+1},$$

an integer number. Here is the demonstration.

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67 **Second step:**
Now, the same thing that has been done with $\frac{M}{P}$, will be done with $\frac{R'}{M}$, but keep in mind that this is not the residue of the continued fraction that defines the tangent, since it is the result of multiplying $\frac{{}^1R}{M}$, which is the residue of said continuous fraction, by φ. This is why the denominator M is multiplied by φ:

$$\frac{R'}{M\cdot\varphi} = \frac{{}^1R\cdot\varphi}{M\cdot\varphi} = \frac{1}{\dfrac{M\cdot\varphi}{{}^1R\cdot\varphi}} = \frac{1}{\dfrac{3\omega}{\varphi} + \dfrac{{}^2R}{R'}},$$

to go back to the continued fraction. Now, ${}^2R$ is the residue of the second division, and the process followed in the step 1 is repeated by multiplying by φ in the appropriate place:

$$\frac{1}{\dfrac{M\varphi^2}{R'}} = \frac{1}{\dfrac{M\varphi\cdot\varphi}{{}^1R\varphi}} = \frac{1}{3\omega + \dfrac{{}^2R\cdot\varphi}{R'}} = \frac{1}{3\omega + \dfrac{R''}{R'}}$$

In this way, $\varphi^2 M = 3\omega R' + R''$, obtaining again two equalities:

$$\begin{cases} R'' = \varphi^2 M - 3\omega R' \\ \\ \dfrac{R''}{D} = r'' \in \mathbb{Z} \end{cases}$$

Once again, the second equality is deduced directly from the first one by dividing by D, and taking into account that both $\frac{M}{D}$ and $\frac{R'}{D}$ are integer numbers.

68 There is an error here, as the expression should be written as follows:

$$R^{n+2} : D = r^{n+2}$$

§. 44. Dividing $\varphi^2 R^n$ by R^{n+1}, the quotient will be[69] $(2n+1)\omega =$ integer number, & being the residue $= R^{n+2}$, we shall have

$$\varphi^2 R^n = (2n+1)\omega \cdot R^{n+1} + R^{n+2},$$

hence dividing by D,

$$\frac{\varphi^2 \cdot R^n}{D} = \frac{(2n+1)\omega \cdot R^{n+1}}{D} + \frac{R^{n+2}}{D}$$

But

$$\frac{\varphi^2 R^n}{D} = \varphi^2 r^n = \text{integer number},$$

$$\frac{(2n+1)\omega \cdot R^{n+1}}{D} = (2n+1)\omega r^{n+1} = \text{integer number},$$

hence

$$\varphi^2 r^n = (2n+1)\omega \cdot r^{n+1} + \frac{R^{n+2}}{D},$$

which gives

$$\frac{R^{n+2}}{D} = \varphi^2 \cdot r^n - (2n+1)\omega \cdot r^{n+1} = \text{integer number} = r^{n+2}.$$

And this is what had to be proven.

§. 45. Now we have seen that r', r'' are integer numbers (§. 41. 42.) therefore also r''', r^{IV}, $\cdots\cdots r^n \cdots\cdots$ &c. to infinity will be integer numbers. Hence, indistinctly, all the residues R', R'', $R''' \cdots\cdot R^n \cdots$ &c. to infinity will have D as common divisor. Let us now find the value of these residues expressed through M, P.

§. 46. For this purpose, each division provides us with an equation, in which

$$\begin{aligned} R' &= \varphi P - \omega M, \\ R'' &= \varphi^2 M - 3\omega \cdot R', \\ R''' &= \varphi^2 R' - 5\omega \cdot R'', \\ &\text{\&c.} \end{aligned}$$

But let us observe that, in the existing case, the quotients ω, 3ω, 5ω &c. are alternately positive & negative, & that the signs of the residues follow each other in the

[69] There is a problem with indices since, for example, if $n = 1$, we would obtain $\varphi^2 R' = 3\omega \cdot R'' + R'''$ instead of the right expression $\varphi^2 R' = 5\omega \cdot R'' + R'''$. The general term for the quotient should therefore be $(2n+3)\omega$ (take this into account throughout this point and at §. 46).

order $- - ++$. Therefore these equations become[70]

$$\begin{aligned} R' &= \omega M - \varphi P, \\ R'' &= 3\omega R' - \varphi^2 M, \\ R''' &= 5\omega R'' - \varphi^2 R', \\ R^{IV} &= 7\omega R''' - \varphi^2 R'', \\ &\text{\&c.} \end{aligned}$$

And in general

$$R^{n+2} = (2n-1)\omega \cdot R^{n+1} - \varphi^2 R^n.$$

From where we see that each residue is obtained, by means of the two preceding ones, in the same way as the numerators & denominators of the fractions that approximate the value of[71] $\tan \frac{\varphi}{\omega}$. (§. 36.)

§. 47. Doing therefore the substitutions that these equations indicate, in order to express all these residues by means of M, P, we shall have

$$\begin{aligned} R' &= \omega M - \varphi P, \\ R'' &= (3\omega^2 - \varphi^2)M - 3\omega\varphi \cdot P, \\ R''' &= (15\omega^3 - 6\omega\varphi^2)M - (15\omega^2\varphi - \varphi^3)P, \\ &\text{\&c.} \end{aligned}$$

And these coefficients of M, P being the numerators & denominators of the fractions found above for the $\tan \frac{\varphi}{\omega}$, (§. 36.), we also see that we shall have

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70 Here is where the signs are taken into account.

71 A succession of real numbers $R', R'', R''', \ldots, R^n, \ldots$, is therefore constructed and defined as follows:

$$\begin{cases} R' = \omega M - \varphi P \\ \\ R'' = 3\omega R' - \varphi^2 M \\ \\ R^{n+2} = (2n+3)\omega R^{n+1)} - \varphi^2 R^{n)}\ , \ n \geq 1 \end{cases}$$

Moreover, we also have a succession of non zero integer numbers $r', r'', r''', \ldots, r^n, \ldots$, connected to the former as follows:

$$R^n = D \cdot r^n\ , \ n \geq 1,$$

with $D \notin \mathbb{Q}$. The last step is to prove that this succession converges to zero (§. 47–§. 51).

$$\frac{M}{P} - \frac{\varphi}{\omega} = \frac{R'}{\omega P},$$

$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{R''}{(3\omega^2 - \varphi^2) \cdot P},$$

$$\frac{M}{P} - \frac{15\omega^2\varphi - \varphi^3}{15\omega^3 - 6\omega\varphi^2} = \frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P},$$

&c.

§. 48. But we have[72]

$$\frac{M}{P} = \tan\varphi.$$

Hence[73] (§. 37. 34.)

$$\frac{M}{P} - \frac{\varphi}{\omega} = \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^3)} + \&c.$$

$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c.$$

&c.

Hence

$$\frac{R'}{\omega P} = \frac{\varphi^3}{\omega(3\omega^2 - \varphi^2)} + \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c.$$

$$\frac{R''}{(3\omega^2 - \varphi^2)P} = \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c.$$

$$\frac{R'''}{(15\omega^3 - 6\omega\varphi^2)P} = \frac{\varphi^7}{(15\omega^3 - 6\omega\varphi^2) \cdot (105\omega^4 - 45\omega^2\varphi^2 + \varphi^4)} + \&c.$$

&c.

Thus all the residues will be found by means of the series of differences (§. 37.)

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[72] There is an error here. The right side of the equality should be written as follows:

$$\frac{M}{P} = \tan\frac{\varphi}{\omega}.$$

[73] There is an error in the first expression. It should be:

$$\frac{M}{P} - \frac{3\omega\varphi}{3\omega^2 - \varphi^2} = \frac{\varphi^5}{(3\omega^2 - \varphi^2) \cdot (15\omega^3 - 6\omega\varphi^2)} + \&c.$$

$$\begin{aligned}\tan\frac{\varphi}{\omega} = \frac{\varphi}{\omega} + \frac{\varphi^3}{\omega(3\omega^2-\varphi^2)} + \frac{\varphi^5}{(3\omega^2-\varphi^2)(15\omega^3-6\omega\varphi^2)} \\ + \frac{\varphi^7}{(15\omega^3-6\omega\varphi^2)(105\omega^4-45\omega^2\varphi^2+\varphi^4)} \\ + \&\text{c.}\end{aligned}$$

by omitting the first terms 1, 2, 3, 4 &c., & multiplying the sum of the following ones by the first factor of the denominator of the first term that is being retained, & by[74] P.

§. 49. Now, this series of differences is more convergent than any decreasing geometric progression (§. 34. 35.). Therefore the residues R', R'', R''' &c. decrease in such a way that they finally become smaller than any assignable quantity.[75] And since each of these residues, having D as a common divisor, is a multiple of D, it follows

[74] In summary, considering that:

$$\frac{M}{P} = \mathcal{S} = \underbrace{\underbrace{\underbrace{\underbrace{\frac{p_0}{q_0}}_{\mathcal{S}_1} + \left(\frac{p_1}{q_1} - \frac{p_0}{q_0}\right)}_{\mathcal{S}_2} + \left(\frac{p_2}{q_2} - \frac{p_1}{q_1}\right)}_{\mathcal{S}_3} + \left(\frac{p_3}{q_3} - \frac{p_2}{q_2}\right)}_{\mathcal{S}_4} + \cdots,$$

where $\mathcal{S}_n$ stands for the nth partial sum of the infinite series $\mathcal{S}$ converging to $\frac{M}{P}$ defined through its continued fraction, we have:

$$\frac{M}{P} - \frac{p_n}{q_n} = \mathcal{S} - \mathcal{S}_n,$$

and so:

$$\frac{R'}{\omega P} = \mathcal{S} - \mathcal{S}_1, \quad \frac{R''}{(3\omega^2-\varphi^2)P} = \mathcal{S} - \mathcal{S}_2, \quad \frac{R'''}{(15\omega^3-6\omega\varphi^2)P} = \mathcal{S} - \mathcal{S}_3 \quad \cdots \quad \frac{R^{n)}}{F_n P} = \mathcal{S} - \mathcal{S}_n \quad \cdots$$

Therefore $R^n = (\mathcal{S} - \mathcal{S}_n)P \cdot F_n$, F_n being the first factor of the denominator of the nth convergent of $\tan\frac{\varphi}{\omega}$, $n \geq 1$.

[75] It is more than likely that the reader is wondering the same question than the author of this work asked himself: Why the fact that $\mathcal{S} - \mathcal{S}_n \xrightarrow[n\to\infty]{} 0$ faster than any decreasing geometric progression, implies that $R^n \xrightarrow[n\to\infty]{} 0$? In the following equality:

$$R^n = (\mathcal{S} - \mathcal{S}_n)P \cdot F_n$$

there is a factor F_n converging to infinity $F_n \xrightarrow[n\to\infty]{} \infty$. Is it sufficiently clear that the convergence of $\mathcal{S} - \mathcal{S}_n$ to zero is fast enough as to overtake F_n and therefore to conclude that $R^n \xrightarrow[n\to\infty]{} 0$? One seems to be led to try to prove that $(\mathcal{S} - \mathcal{S}_n)$ (A_n to simplify the writing; P does not affect since it is a constant) effectively converges faster than any decreasing geometric progression and that F_n does not, that is to say, that F_n converges more slowly than an increasing geometric progression, a reasoning that seems to be implicit in Lambert's statement (although this approach would not lead us anywhere, as we will see). What is then the situation? In the light of later results on continued fractions, Christopher Baltus in Baltus (2003) analyzes the case (I thank Professor Baltus for his that this common divisor D is smaller than any assignable quantity, which makes $D = 0$, & leads to the consequence that $(M : P)$ is a quantity incommensurable to unity, or irrational.[76]

§. 50. Hence *every time an arc of a circle* $= \frac{\varphi}{\omega}$ *is commensurable to the radius* $=1$, *or rational, the tangent of this arc will be a quantity incommensurable to the radius, or irrational*. And conversely *no rational tangent is that of a rational arc*.[77]

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kindness, his reflection on the subject, and for having facilitated to me the work with which I was able to better understand this issue). Broadly speaking, he studies how fast the three intervening successions converge, and he does it by analyzing the ratios between consecutive terms (term by term, since each A_n is itself an infinite series, although we will be setting out here the situation in global terms). The relationship between these ratios is:

$$\frac{R^{n+1}}{R^n} = \frac{A_{n+1}}{A_n} \cdot \frac{F_{n+1}}{F_n}$$

from where, as he proves:

$$\frac{A_{n+1}}{A_n} \xrightarrow[n\to\infty]{} 0 \quad \text{y} \quad \frac{F_{n+1}}{F_n} \xrightarrow[n\to\infty]{} \infty,$$

and where:

$$\frac{R^{n+1}}{R^n} \xrightarrow[n\to\infty]{} 0,$$

which implies:

$$R^n \xrightarrow[n\to\infty]{} 0$$

What these last four relations are telling to us, is that despite the fact that the succession F_n converges to infinity (and fast, in fact), the succession R^n converges to zero given the «supremacy of A_n over F_n», just as Lambert had claimed. Now, going back to what Lambert seems to say, the statement:

R^n converges to zero becasue $A_n \xrightarrow[n\to\infty]{} 0$ faster than any decreasing geometric progression

seems to be misleading, since $\frac{F_{n+1}}{F_n} \xrightarrow[n\to\infty]{} \infty$, that is to say, the succession F_n does not converge to infinity slower than an increasing geometric progression; it does faster, since given such a progression:

$$a,\ a^2,\ a^3, \ldots, a^n,\ a^{n+1}, \ldots$$

with $|a| > 1, a \neq 0$:

$$\frac{a^{n+1}}{a^n} \xrightarrow[n\to\infty]{} a < \infty$$

It is not clear for us how to interpret Lambert's words.

[76] That is to say, since:

$$R^n = D \cdot r^n$$
$$R^n = (\mathcal{S} - \mathcal{S}_n)P \cdot F_n,$$

and that, according to what Lambert affirms and that has just been discussed, $R^n = (\mathcal{S} - \mathcal{S}_n)P \cdot F_n \xrightarrow[n\to\infty]{} 0$, there follows as a necessary consequence that $D = 0$ due to the fact that all r^n are non zero integer numbers, which is a contradiction because $D \notin \mathbb{Q}$.

[77] Two points should be made here. The first one is that this result was already announced in 1719 in *Mémoire sur la quadrature du cercle, & sur la mesure de tout Arc, tout Secteur, & tout Segment*

§. 51. Now, the tangent of 45° being rational, equal to the radius, it follows that the arc of 45° degrees, & therefore also the arc of 90°, 180°, 360° degrees, is incommensurable to the radius. Therefore *the circumference of the circle is not to the diameter as an integer number to an integer number*. So here is this theorem in the form of a corollary of another infinitely more universal theorem.[78]

§. 52. Indeed, it is precisely this absolute universality that may well surprise us. In addition to letting us know to what extent the circular quantities are transcendental, it also shows us, *that the rational tangents & the rational arcs are not distributed throughout the circumference of the circle, as if they were placed at random, but that there has to be a certain order, & that this order prevents them from ever meeting*. This order certainly deserves, sans contredit, to be known in more detail. Hence, let us see how far it will be possible to determine the laws. This is what the following theorems will lead to.[79]

§. 53. First of all, we know that *if two tangents are rational, the tangent of the sum & that of the difference of their arcs are equally rational*. Since[80]

$$\text{tang}\,(\omega + \varphi) = \frac{t\omega + t\varphi}{1 - t\omega \cdot t\varphi},$$

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donné by Thomas Fantet de Lagny (De Lagny 1719/1721), although with no proof (Brezinski 1991, pp. 95, 96). I do not know if Lambert was aware of this fact, although I would lean towards he was not, since Lambert does not seem to have had any problems when citing sources —he quotes Foncenex (see §. 74.) and he also acknowledges that Euler drew him into the study of continued fractions— and Lagny's name does not appear. It is true that Lambert in Lambert (1766/1770, p. 146) refers to the 127 decimal places of the approximation of π given by Lagny precisely in this work, but it is possible that he was only echoing a fact that, on the other hand, was widely known. The second thing to mention is that the inverse of the theorem proved by Lambert does not hold, that is:

$$\tan v \notin \mathbb{Q} \quad \not\Longrightarrow \quad v \in \mathbb{Q},$$

since there are irrational arcs with irrational tangents. In fact, every arc of the form $v = \frac{k\pi}{n} \notin \mathbb{Q}$ satisfies that $\tan v \notin \mathbb{Q}$ (except for $v = \pm\pi, \pm\frac{\pi}{4}$) (Calcut 2006).

[78] In a nutshell: since $\tan \frac{\pi}{4} = 1 \in \mathbb{Q}$, then $\frac{\pi}{4} \notin \mathbb{Q}$, and so:

$$\pi \notin \mathbb{Q}$$

Here we have the first demonstration of the irrationality of π.

[79] Lambert begins to extract —from §. 53 to §.71— a series of properties for the tangent that end up justifying the introduction of the concept of *prime tangent* (§. 60). It seems that he takes these steps to somehow show «...what reduces infinitely the possibility of finding a rational arc, whose tangent is equally rational» (§. 68), that is, to show in a heuristic way, the fact that the tangent of a rational arc cannot be rational, which is what he has just been demonstrated in the first part of this work. These same properties he extracts for the tangent can be extended to cosine (§. 70) but not to sine (§. 71).

[80] Note from now on that Lambert writes «tang» or «t» interchangeably to refer to the tangent.

$$\text{tang}\,(\omega - \varphi) = \frac{t\omega - t\varphi}{1 + t\omega \cdot t\varphi}.$$

§. 54. From this it follows *that if a tangent is rational, the tangent of any multiple of its arc will be equally rational.*[81]

§. 55. But conversely, *if a tangent is rational,*[82] *no aliquot part of its arc will have a rational tangent*. Because being the proposed arc a multiple of each of its aliquot parts, it is clear that if the tangent of one of its aliquot parts were rational, the tangent would be rational (§. 54.).

§. 56. *If the tangent of each of two commensurable arcs to one another is rational, the tangent of the greatest common divisor of these two arcs will be equally rational.* Let ω, φ be the two proposed arcs. Now, being commensurable, ω will be to φ as an integer number m to an integer number n. Let these numbers m, n, be prime to one another, & the unit will be their greatest common divisor. Therefore doing

$$\omega = m\psi,$$

$$\varphi = n\psi,$$

& the arc ψ will be the greatest common divisor of the arcs ω, φ. Now I say that the tang ψ will be rational. Let $m > n$, & subtracting n from m as many times as possible, let the last remainder be $= r$, all the[83] tang $(m - n)\psi = t(\omega - \varphi)$, tang $(m - 2n\psi) = t(\omega - 2\varphi)$, &c. tang $r\psi$, will be rational (§. 53.). Subtract r from n as many times as possible, let the last residue be $= r'$. Now subtract r' from r as many times as possible, let the last residue be r'' &c. And continuing in this way, the numbers m, n being prime to one another, you will arrive at a residue $= 1$ (*Euclid.* Pr. I. Livr. VII.). But by §. 53. all the tangents[84]

$$\begin{array}{llcl}
t(m-n)\psi, & t(m-2n)\psi & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & tr\psi, \\
t(n-r)\psi, & t(n-2r)\psi & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & tr'\psi, \\
t(r-r')\psi, & t(n-2r')\psi & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & tr''\psi, \\
\text{\&c.} & & & \\
\cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot & t\psi,
\end{array}$$

will be rational. Hence &c.

[81] Lambert is thinking of integer multiples (see §. 62).

[82] See the note by A.S. in Appendix C. It is clear, as is easily deduced from the reasoning exposed by Lambert in the following lines, that there is an error here: it should be written «irrational».

[83] There is an error here. As can be easily verified, it should be written as follows:

$$\text{tang}\,(m-n)\psi = t(\omega - \varphi),\ \text{tang}\,(m-2n)\psi = t(\omega - 2\varphi),\ \text{\&c. tang}\,r\psi$$

[84] There is an error in the second column of the third row. It should be: $t(r - 2r')\psi$.

§. 57. It is clear that in this way (§. 53.), given any two rational tangents, all these tangents can be known by means of tang ω, tang φ, without the arcs being known, will we find if their arcs are commensurable with each other? But if the arcs are not, the task will never end.

§. 58. *If any two aliquot parts of any arc have rational tangents, I say that the tangent of the greatest common divisor of these two aliquot parts will be equally rational*. This theorem follows immediately from the preceding one (§. 56.). One needs only to remember that two arcs ω, φ, which are aliquot parts of an arc A, are commensurable with each other.

§. 59. Similarly, *if as many aliquot parts of an arc A as we want have rational tangents, the tangent of the arc, which is the greatest common divisor of these aliquot parts, will be equally rational*. Take two of these aliquot parts ω, φ, & let their greatest common divisor be $= \psi$, & the tang ψ will be rational (§., 56. 58.). But ψ being an aliquot part of the arcs ω, φ, which are aliquot parts of the arc A, it is clear that ψ will be an aliquot part of the arc A, & that in the place of the arcs ω, φ, we can substitute ψ by comparing ψ with one of the other aliquot parts proposed of the arc A. We shall continue to find their greatest common divisor, the tangent of which will be equally rational. &c.

§. 60. Let us call *prime tangent* every rational tangent, which is that of an arc, of which no aliquot part has a rational tangent.

§. 61. Such is for example the tangent of 45°. Since, let n be any integer number, all tang $(45 : n)^\circ$ will be one of the roots of the equation

$$0 = 1 - nx - n \cdot \frac{n-1}{2} x^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} x^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot x^4 \\ - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} x^5 - \&c.$$

whose coefficients are the same as those of the *Newton* binomial formula, & whose signs change following the order $- - ++$. But, for every integer number n, all these coefficients are integer numbers, & every

$$\text{tang}\left(\frac{45^\circ}{n}\right) < 1.$$

Therefore, if one or more than one of the tang $(45^\circ : n)$ were rational, it would be a rational *fraction* < 1, & if so, not all the coefficients could be integer numbers. But they are. Therefore &c.[85]

[85] Let us take the formula for the tangent of the multiple angle as reference:

§. 62. *Let any prime tangent be given, only the multiples of its arc have rational tangents, excluding all the arcs that are commensurable to it.* Let tang ω be prime, & m, n, being integer prime numbers to one another, let us suppose that the tang $\left(\frac{m}{n}\omega\right)$ could be rational. Now the arc $\left(\frac{\omega}{n}\right)$ being the greatest common divisor of the arcs ω, & $\left(\frac{m\omega}{n}\right)$, the tangent of $\frac{\omega}{n}$ will be rational (§. 56.). But $\frac{\omega}{n}$ being an aliquot of ω, the tang ω would not be prime. This being contrary to the hypothesis, we see that no tang $\left(\frac{m}{n}\omega\right)$ will be rational. Therefore, only the multiples of ω remain, the tangents of which will be rational (§. 54.).[86] Here is the reason why this type of tangents deserve the name of primes. They somehow resemble prime numbers, insomuch as only their multiples are integer numbers, &c.

$$\tan na = \frac{\sum_{k=0}^{\frac{n-1}{2}} (-1)^k \binom{n}{2k+1} \tan^{2k+1} a}{\sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n}{2k} \tan^{2k} a} =$$

$$= \frac{\binom{n}{1}\tan a - \binom{n}{3}\tan^3 a + \binom{n}{5}\tan^5 a - \binom{n}{7}\tan^7 a + \cdots}{1 - \binom{n}{2}\tan^2 a + \binom{n}{4}\tan^4 a - \binom{n}{6}\tan^6 a + \binom{n}{8}\tan^8 a + \cdots}$$

The reason why the combinatorial numbers of Newton's binomial appear (as Lambert mentions), is that this formula is deduced from Moivre's formula:

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha,$$

expanding the term on the left (by using Newton's binomial), equating the real and imaginary parts, and dividing conveniently (Maor 2013, pp. 154–155). Putting $a = \frac{45}{n}$, we have that $\tan na = 1$, so the numerator and denominator of the expression above must to be equal one another:

$$\binom{n}{1}\tan a - \binom{n}{3}\tan^3 a + \binom{n}{5}\tan^5 a - \cdots = 1 - \binom{n}{2}\tan^2 a + \binom{n}{4}\tan^4 a - \cdots$$

Passing everything to the first member, we obtain the equation that Lambert writes, without more than putting $x = \tan a$. As for why Lambert concludes that $\tan(45 : n)$ cannot be a root of that equation since all the coefficients are integers, it is something that can be concluded from what is called the «rational root theorem» —actually an elementary result—, which imposes restrictions on the rational roots of polynomials with integer coefficients. Let a polynomial be given:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

with integer coefficients: if $\frac{p}{q}$ (with p and q being coprime each other) is a root, then p is divisor of the independent term a_0 and q is divisor of the main term a_n. But in the case discussed by Lambert, $a_0 = a_n = 1$, so that fraction should be equal to 1, which is not possible since our fraction $\frac{p}{q}$ is nothing else, by hypothesis, than $\tan(45 : n) < 1$.

[86] That is to say: we knew (§. 54) that integer multiples gave rational tangents, and now we know that —if the tangent is prime— this does not happen with submultiples (in fact with rational multiples in general). This similarity with prime numbers is what leads Lambert to call them *prime* tangents.

§. 63. *Let two prime tangents be given, I say that their arcs are incommensurable with each other*. Since tang ω, tang φ are primes, & let us suppose the arcs ω, φ could be commensurable with each other. They will therefore be as an integer number m to an integer number n. Hence

$$\varphi = \frac{m\omega}{n}.$$

Hence (§. 62.)[87] $\frac{\omega}{n}$, aliquot part of ω, will have a rational tangent, likewise $\frac{\varphi}{m}$ aliquot part of φ. Therefore $t\varphi$, $t\omega$, will not be primes. This being contrary to the hypothesis, it is clear that the arcs ω, φ cannot be commensurable with each other.

§. 64. *Thus all the arcs of prime tangents are incommensurable with each other*. Since, by the preceding theorem, they are two by two, combined in any way.

§. 65. *Let any rational tangent that is not prime be given, I say that its arc will be a multiple of that of a prime tangent*. Since this tangent, as rational as it is, is not prime, there has to be aliquot parts of its arc the tangents of which are rational. Let these aliquot parts be $\frac{\omega}{m}$, $\frac{\omega}{n}$, $\frac{\omega}{p}$, $\frac{\omega}{q}$ &c. which are assumed to be finite. Now, since we take them all, it is necessary that the one that is the greatest common divisor of all the others is also among them, while by §. 59. the tangent is equally rational. Let it be $\frac{\omega}{r}$, I say tang $\frac{\omega}{r}$ is prime. Since, if it were not prime, the tangents of some of the aliquot parts of $\left(\frac{\omega}{r}\right)$ would be rational. Now, since these aliquot parts of $\left(\frac{\omega}{r}\right)$ are equally aliquot parts of the proposed arc ω, it is clear that they would already be included in the aliquot parts $\frac{\omega}{m}$, $\frac{\omega}{n}$, $\frac{\omega}{p}$ $\cdots\cdot$ $\frac{\omega}{r}$, & that therefore $\frac{\omega}{r}$ would equally be its greatest common divisor. Thus $\frac{\omega}{r}$ would be measure of its aliquot parts. This being absurd, we see that tang $\frac{\omega}{r}$ is prime. Now ω is a multiple of $\frac{\omega}{r}$. Therefore &c.

§. 67.[88] Here are therefore *all rational tangents ordered in certain classes*. Either they are themselves prime, or they descend, so to speak, in a straight line from a prime tangent, since only the multiples of arcs with prime tangents have rational tangents (§. 62.). But,[89] if there were only one prime tangent, all rational tangents would follow from it, & all their arcs would be commensurable with each other. But it is far

[87] This reference to the point §. 62 alludes, not to the main result, which is the one in italics, but to the first part of the reasoning contained therein. Specifically, since the tangents of $\frac{m\omega}{n}$ $(= \varphi)$ and ω are rational by hypothesis, that of $\frac{\omega}{n}$ is also rational because it is the greatest common divisor (§. 56) (analogously for φ).

[88] There is an error in the numbering. It should be «§. 66.».

[89] Lambert then proves that there are infinitely many prime tangents. Lambert says that if there was only one ($\tan\varphi$) it would be smaller than any assignable quantity because as he will explain, there is always a prime tangent below —in magnitude— than another previously fixed one. This not just proves that there is not only one —which is what he starts the following lines with—, but that there are **infinite** (see next footnote).

from being the case that there is only one prime tangent. Since it should be smaller than any assignable quantity. Let us give it, to prove this, a finite magnitude $= \tan\varphi$. It is clear that there will be rational tangents smaller than $\tan\varphi$. If these tangents are prime, $\tan\varphi$ will not be the only one that is prime. If they are not prime, they will derive from one or several prime tangents, inasmuch as their arcs will be multiples of those of these prime tangents (§. 65.). Thus there is more than one, more than 2, 3, 4, &c. prime tangents. And whenever a finite number is supposed, it will be found in the same way that there are more. Here is another way to find an infinite number.[90]

§. 67. Let two prime tangents be $t\omega$, $t\varphi$. Firstly they will be rational, & their arcs will be incommensurable with each other (§. 64.). Let m, n any numbers prime to one another be given, & $(m\omega + n\varphi)$ will be an arc incommensurable to both ω and φ. Now the tangent will be rational (§. 62. 53.).[91] But the arc $(m\omega + n\varphi)$ not being multiple of either ω or φ, the tang $(m\omega + n\varphi)$ will either be itself prime, or it will derive from a prime tangent, necessarily different from $t\omega$, $t\varphi$. But, by varying the numbers m, n, in all possible ways, so that they are always prime to one another, we shall find as many arcs $(m\omega + n\varphi)$ incommensurable both among themselves and among the arcs ω, φ, & that consequently are neither multiples of each other, nor of ω, φ. Therefore, their tangents, which are all rational, will derive from as many prime tangents, different from each other.

§. 68. Here is therefore what reduces infinitely the possibility of finding a rational arc, whose tangent is equally rational. Since the arcs of all the prime tangents are incommensurable with each other, it follows that, when it is possible to find a prime tangent whose arc is commensurable to the radius, this will be the only one, since the arcs of all the other prime tangents would necessarily be incommensurable to the radius.[92] But, from what we have seen above,[93] this only one is also excluded from the possibility of having a rational arc.

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[90] This other way is actually quite old, at least as old as Euclid's *Elements*, where the author proves by using the same reasoning that «prime numbers are more than any assigned multitude of primer numbers» (Heath II 1908, p. 412). Euclid is very careful with the language and avoids any reference to actual infinity (avoiding affirmations of the style «there are infinitely many prime numbers»), taboo in the Greek world, but although the format this reasoning takes in the hands of Lambert is the same («And whenever a finite number is supposed, it will be found in the same way that there are more»), for the Swiss it means «...another way to find an infinite number». It cannot be stated with certainty, but it seems to indicate not only Lambert's acceptance of actual infinity, but also the fact that for him a potential infinity already presupposes an actual infinity. Both things are not trivial if we take into account the time. Cantor will insist on this idea a century later (see Bermúdez (2009, pp. 432, 448, 450, 462, 469)). In addition, it is worth mentioning that in the fourth part of *Anlage zur Architektonik* Lambert devoted an extensive section to the concept of infinity cf. Lambert (1771, pp. 547ff.). I am grateful to one of the anonymous reviewers for bringing this point to my attention.

[91] Actually the fact that it is rational is only due to the tangents of ω and φ being rational, as established in §. 53. The reference to §. 62 seems to be superfluous.

[92] This is because otherwise they would be commensurable with each other, which contradicts §. 64.

[93] He refers to §. 50.

§. 69. The tangent of 45° being prime (§. 61.) & being found in the trigonometric tables,[94] I will further note as a corollary that this is the only prime tangent, & at the same time the only rational tangent found there. The reason is that all the arcs whose tangents are indicated in these tables are commensurable with each other, without another multiple of 45° being found there other than the angle of 90°, whose tangent is infinite.

§. 70. I shall also observe that, the cosine of any angle ω being rational, the cosine of any multiple is equally rational.[95] This circumstance makes that the same reasoning that we have exposed in relation to the tangents, can be applied with a few changes to the cosines. *Prime cosines* will be found as we have found *prime tangents*, & the arcs of the prime cosines will be equally incommensurable with each other; so that, when it is possible to find a prime cosine whose arc is rational, this would still be the only one that could be found, since, similarly, the arcs of all the other prime cosines would be irrational.

§. 71. This is not the same for the sines[96] since, given any rational $\sin\omega$, there are in general no more rational sines than $\int 3\omega$, $\int 5\omega$, $\int 7\omega$ &c.; but the $\sin 2\omega$, $\int 4\omega$, $\int 6\omega$ &c. are not always so, unless $\cos\omega$ is also rational, thus if one wants to find *prime sines* here as well, one will have to do it in another way from the one we have done in relation to the tangents.

§. 72.[97] But, without dwelling on it, I will return to the continued fraction found above

$$\tan v = \cfrac{1}{\omega - \cfrac{1}{3\omega - \cfrac{1}{5\omega - \cfrac{1}{7\omega - \cfrac{1}{9\omega - 1 \text{ \&c.}}}}}}$$

We have seen that all the fractions

$$\frac{1}{\omega}, \quad \frac{3\omega}{3\omega^2 - 1}, \quad \frac{15\omega^2 - 1}{15\omega^3 - 6\omega}, \quad \text{\&c.}$$

[94] Lambert may be referring to his trigonometric tables included in Lambert (1768/1770).

[95] This follows from the multiple angle formula for cosine. The reasoning applied here to the cosine cannot be transferred —as explained in the next point (§. 71)— to the sine. Without going into details, everything follows from the formulas of the multiple angle, which, by the way, were known since Viète Boyer (1968, p. 393).

[96] Lambert interchangeably uses «sin» and «$\int$» to represent the sine.

[97] Lambert makes here some remarks on the convergents of the continued fraction for $\tan v$, and obtains the continued fraction for the $\cot v$.

that it provides do not approximate the value of the tangent of v but by default, inasmuch as they are all smaller than this tangent. But, since it must be possible to find similar fractions, which, approaching the tangent, do so by excess, I set out to investigate it. Here again I will confine myself to giving the continued fraction containing alternately & the ones & the others. Here it is[98]

$$\tan v = \cfrac{1}{0 + \cfrac{1}{(\omega - 1) + \cfrac{1}{1 + \cfrac{1}{(3\omega - 2) + \cfrac{1}{1 + \cfrac{1}{(5\omega - 2) + \cfrac{1}{1 + \ \&c.}}}}}}}$$

This fraction continues to infinity, so that the quotients are

$$0, \quad (\omega - 1), \quad 1, \quad (3\omega - 2), \quad 1, \quad (5\omega - 2), \quad 1, \quad (7\omega - 2), \quad 1, \quad (9\omega - 2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad 1, \quad ((2n + 1)\omega - 2), \quad 1 \quad \&c.$$

And the fractions that approximate the value of the tang v are

$$\frac{1}{\omega - 1}, \quad \frac{1}{\omega}, \quad \frac{3\omega - 1}{3\omega^2 - \omega - 1}, \quad \frac{3\omega}{3\omega^2 - 1}, \quad \frac{15\omega^2 - 3\omega - 1}{15\omega^3 - 3\omega^2 - 6\omega + 1},$$

$$\frac{15\omega^2 - 1}{15\omega^3 - 6\omega}, \quad \&c.$$

The first, 3rd, 5th, 7th &c. ones are bigger than tang v, & the 2nd, 4th, 6th &c, ones are smaller, & the same as those we found above (§. 22.). I will not stop to give the proof, since this continued fraction can be found in the same way that we have found the one we have used so far, & which is much simpler. I will only observe, hence, that the first quotient here being $= 0$, in order to eliminate it we shall do nothing but to turn the fraction upside down so that it expresses the cotangent of v, since

$$\cot v = \frac{1}{\text{tang}\, v}.$$

[98] See the note by A.S. in Appendix C.

Thus we shall have

$$\cot v = \cfrac{1}{(\omega-1)+\cfrac{1}{1+\cfrac{1}{(3\omega-2)+\cfrac{1}{1+\cfrac{1}{(5\omega-2)+\cfrac{1}{1+\cfrac{1}{(7\omega-2)+\cfrac{1}{1+\ \&\text{c.}}}}}}}}}$$

§. 73. Let us now compare *the circular transcendental quantities with the logarithmic quantities* that are analogous to them.[99] Let e be the number whose hyperbolic logarithm[100] is $= 1$. And we know that if in the two series that we have used above (§. 4.)

$$\sin v = v - \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^7 + \&c.$$

$$\cos v = 1 - \frac{1}{2}v^2 + \frac{1}{2\cdot 3\cdot 4}v^4 - \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}v^6 + \&c.$$

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[99] It is from this moment and up to the point §. 78 that Lambert analyzes the connection between the circular and hyperbolic trigonometric functions (although he will continue to show this connection in the following points), functions that actually acquire these names in the *Opuscula ad res physicas et mathematicas pertinentium* (1757–1762) by Vincenzo de Riccati (Lambert will use these names for the first time in *Observations trigonometriques* of 1768). According to Lambert, such relationship must exist, not only because of the similarity between their series expansions, which he shows at this very point, but also because a simple change of variable, namely, $u = v\sqrt{-1}$, allows us passing from one to another (§. 74). «But here the issue is to show how far this affinity can be developed independently of the imaginary quantities» (§. 75), and such affinity comes from the fact that circular trigonometric functions parameterize the circumference and hyperbolic trigonometric functions parameterize the equilateral hyperbola (§. 78), an idea he credits to «Mr. *de Foncenex*» (§. 74). This is demonstrated in §. 78 (for more details I refer to Barnett (2004), who will be quoted again in the following points). Furthermore, using the similarity between a certain continued fraction obtained at this very point §. 73 with that for tan v, he concludes by applying some properties of continued fractions that he does not make explicit, that «x and e^x will never be two rational quantities at the same time» (§. 74), that is:

$$x \in \mathbb{Q} \Rightarrow e^x \notin \mathbb{Q},$$

generalizing the result that Euler had first obtained in 1737/1744 ($e \notin \mathbb{Q}$) Euler (1744). Lambert mentions this again in §. 81.

[100] By hyperbolic logarithm he is referring to what we call natural logarithm. Euler makes this clear in Chap. 7 of his *Introductio in Analysin Infinitorum*:

> When this base is chosen [e], the logarithms are called natural or hyperbolic. The latter name is used since the quadrature of a hyperbola can be expressed through these logarithms.

(Euler I 1748, p. 97).

all signs are taken to be positive, they become

$$\frac{e^v - e^{-v}}{2} = v + \frac{1}{2\cdot 3}v^3 + \frac{1}{2\cdot 3\cdot 4\cdot 5}v^5 + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}v^7 + \&c.$$

$$\frac{e^v + e^{-v}}{2} = 1 + \frac{1}{2}v^2 + \frac{1}{2\cdot 3\cdot 4}v^4 + \frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}v^6 + \&c.$$

Now treating these last two series in the same way that we have treated the first two (§. 4. & suiv.[101]) the operation will only differ in the signs, which for the present case will all be positive. Since we can easily convince ourselves of this, I will not get into details. We hence shall have[102]

$$\frac{e^v - e^{-v}}{e^v + e^{-v}} = \cfrac{1}{1:v + \cfrac{1}{3:v + \cfrac{1}{5:v + \cfrac{1}{7:v + \cfrac{1}{9:v + \cfrac{1}{11:v + \cfrac{1}{13:v\ \&c.}}}}}}}$$

§. 74. And since

$$\frac{e^v - e^{-v}}{e^v + e^{-v}} = \frac{e^{2v} - 1}{e^{2v} + 1},$$

we see that doing $2v = x$, we shall have

$$\frac{e^x - 1}{e^x + 1} = \cfrac{1}{2:x + \cfrac{1}{6:x + \cfrac{1}{10:x + \cfrac{1}{14:x + \cfrac{1}{18:x + \&c.}}}}}$$

from where we draw

[101] Abbreviation of «suivants» (next, following).

[102] See the note by A.S. in Appendix C.

$$\frac{e^x+1}{2} = \cfrac{1}{1-\cfrac{1}{2:x+\cfrac{1}{6:x+\cfrac{1}{10:x+\cfrac{1}{14:x+\ \&c.}}}}}$$

or[103]

$$\frac{e^x-1}{2} = \cfrac{1}{(2:x)1+\cfrac{1}{6:x+\cfrac{1}{10:x+\cfrac{1}{14:x+\cfrac{1}{18:x+\ \&c.}}}}}$$

We see that these expressions offer consequences similar to those that we have deduced above from the formula

$$\text{tang}\, v = \cfrac{1}{\omega-\cfrac{1}{3\omega-\cfrac{1}{5\omega-\&c.}}}$$

We shall also found here that v & e^v, like x & e^x, will never be rational quantities at the same time. Thus I will not stop to reiterate the deduction. The issue is rather to interpretate the formulas that we have just exposed. I observe hence that they must have, in relation to the equilateral hyperbola, a totally analogous meaning to that of the fraction

$$\text{tang}\, v = \cfrac{1}{\omega-\cfrac{1}{3\omega-\&c.}}$$

in relation to the circle. Since, in addition to knowing that the expressions

[103] There is an error here. It should be written as follows:

$$\frac{e^x-1}{2} = \cfrac{1}{(2:x)-1+\cfrac{1}{6:x+\cfrac{1}{10:x+\cfrac{1}{14:x+\cfrac{1}{18:x+\ \&c.}}}}}$$

$$e^u + e^{-u}$$

$$e^u - e^{-u}$$

by doing $u = v\sqrt{-1}$, yield the circular quantities[104]

$$\begin{aligned} e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} &= 2\cos v, \\ e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} &= 2\sin v \cdot \sqrt{-1}. \end{aligned}$$

Mr. *de Foncenex* has also shown, in a very simple & very direct way, how this affinity is found by simultaneously comparing the circle & the equilateral hyperbola which have the same center & the same diameter. See *Miscell. Society. Taurin.* Tom. I.p. 128. suiv.[105]

§. 75. But here the issue is to show how far this affinity can be developed independently of the imaginary quantities.[106] Let therefore be C the center, CH the axis, CA the semi-diameter of the equilateral hyperbola AMG & of the circle AND, CF the asymptote, AB perpendicular to the axis, & at the same time the common

[104] There is an error. It should be written as follows:

$$\begin{aligned} e^{v\sqrt{-1}} + e^{-v\sqrt{-1}} &= 2\cos v \\ e^{v\sqrt{-1}} - e^{-v\sqrt{-1}} &= 2\sin v \cdot \sqrt{-1}. \end{aligned}$$

[105] The referred article is *Réflexions sur les quantités imaginaries*.

[106] Here begins the proof of the aforementioned analogy. In this regard, it is necessary to make a few comments. First of all, Lambert includes a figure on the last page with all the data he deals with in the text. Secondly, concerning the notation, when he writes, for example, $\cot \varphi^2$, he is actually referring to $(\cot \varphi)^2$. Finally, something that he comments later, almost at the end of the point §. 77 but that it would be helpful to take into account from the beginning: the argument of the circular functions, the (circular) angle, is defined through the circumference. Whether it is measured in degrees or radians (let us think of radians), its value depends on the circumference. What would be the analogous in the case of the hyperbola? The idea is to change the argument of the circular trigonometric functions and make them depend on the area defined by the angle and the circumference (ANCA in Lambert's drawing) instead of the angle (MCA$= \varphi$). If we represent this angle by v and, as said before, we think in radians, the arc of circumference that defines will measure $v \cdot r = v$ (Lambert thinks of a circle with radius 1). This being the case, the area of the circular sector determined by said angle will be:

$$ANCA = \frac{vr^2}{2} = \frac{v}{2} \quad , \quad \text{from where} \quad v = 2 \cdot ANCA$$

In this way, the argument of the c.t.f. is twice the area of the circular sector determined by the points on the circumference, and therefore in complete analogy, the argument of the h.t.f. will be twice the area of the hyperbolic sector determined by the points on the hyperbola:

$$u = 2 \cdot AMCA$$

tangent to the circle & to the hyperbola. Draw from the center C two straight lines CM, Cm, infinitely close to each other, & lower the points of intersection M, m, N, n, on the ordinate axis MP, mp, NQ, nq. Finally let the radius be AC $= 1$. Let us do the angle MCA $= \varphi$, & be[107]

for the hyperbola	for the circle
the abscissa CP$= \xi \cdot \quad \cdot \quad \cdot$	$\cdots$ CD$= x$
the ordinate PM$= \eta \cdot \quad \cdot \quad \cdot$	$\cdots$ QN$= y$,
the segment AMCA$= u : 2 \cdot \quad \cdot$	$\cdot\cdot$ ANCA$= v : 2$
& will be	
$\mathrm{tang}\,\varphi = \dfrac{\eta}{\xi} \quad \cdot \quad \cdot \quad \cdot$	$\cdot\cdot\ \mathrm{tang}\,\varphi = \dfrac{y}{x}$,
$1 + \eta\eta = \xi\xi = \eta\eta \cdot \cot\varphi^2 \quad \cdot$	$\cdot 1 - yy = xx = yy \cdot \cot\varphi^2$,
$\xi\xi - 1 = \eta\eta = \xi\xi \cdot \mathrm{tang}\,\varphi^2 \quad \cdot$	$\cdot 1 - xx = yy = xx\ \mathrm{tang}\,\varphi^2$,
$\mathrm{CM}^2 = \xi^2 + \eta^2 = \xi^2(1 + \mathrm{tang}\,\varphi^2)$	$\cdot \mathrm{CN}^2 = x^2 + y^2 = x^2(1 + t\varphi^2)$,
$= \dfrac{1 + t\varphi^2}{1 - t\varphi^2}$	$= \dfrac{1 + t\varphi^2}{1 + t\varphi^2}=1.$
Hence	
$+du = d\varphi \cdot \left(\dfrac{1 + t\varphi^2}{1 - t\varphi^2}\right) = \dfrac{dt\varphi}{1 - t\varphi^2}$	$\cdot + dv = d\varphi = \dfrac{dt\varphi}{1 + t\varphi^2}$,
$+d\xi = \dfrac{t\varphi \cdot dt\varphi}{(1 - t\varphi^2)^{3:2}} \quad \cdot \quad \cdot \quad \cdot$	$\cdot - dx = \dfrac{t\varphi \cdot dt\varphi}{(1 + t\varphi)^{3:2}}$,
$+d\eta = \dfrac{dt\varphi}{(1 - t\varphi^2)^{3:2}} \cdots$	$\cdot + dy = \dfrac{dt\varphi}{(1 + t\varphi)^{3:2}}$,
$\xi = \dfrac{1}{\sqrt{(1 - t\varphi^2)}} \cdots$	$\cdot\cdot\ x = \dfrac{1}{\sqrt{1 + t\varphi^2}}$,
$\eta = \dfrac{t\varphi}{\sqrt{(1 - t\varphi^2)}} \cdots$	$\cdot\cdot\ y = \dfrac{t\varphi}{\sqrt{1 + t\varphi^2}}$,
Hence	
$+d\xi : du = \eta \quad \cdots\cdot$	$\cdot - dx : dv = y$,
$+d\eta : du = \xi \quad \cdots\cdot$	$\cdot + dy : dv = x$,
$+d\xi = d\eta \cdot \mathrm{tang}\,\varphi \quad \cdot\cdot$	$\cdot - dx : dy = \mathrm{tang}\,\varphi$.

§. 76.[108] Since the angle φ is the same for the hyperbola & for the circle, it follows from the last two equations that

$$\mathrm{tang}\,\varphi = d\xi : d\eta = -dx : dy = \eta : \xi = y : x.$$

Thereby the angles Mmp, Nnq, are equal. Which gives

$$\mathrm{M}m : \mathrm{N}n = d\xi : -dx = d\eta : dy.$$

[107] Lambert does not specify any of the calculations included in the table. Some are straightforward while others require more elaboration (see Barnett (2004, p. 22) for some example), as well as awareness on the part of the modern reader about the use and interpretation of differentials at the time. For instance, the expression $\dfrac{d\xi}{d\eta}$ (see §. 76.) represents for Lambert a quotient, which is what the definition of differential actually corresponds to in the Leibnizian calculus (and from where the modern notation to denote the derivative arises).

[108] At this point, some similarities between the circle and the hyperbola are noted.

And the characteristic triangles M$m\mu$, N$n\nu$, are similar.[109] Finally, since Cnq = Cmp, & Nnq = Mmp, we shall have Cnq + Nnq = Cmp + Mmp = 90°. Hence by drawing the normal mV, we shall have[110] Vmq + Mmq = 90°, hence Vmq = Cmq. Thus the normal mV prolonged to the axis AC is equal to Cm, just as in the circle the normal Cn is equal to Cn.[111] Here is therefore that upon which everything that is real in the comparisons that we have made between the circle & the hyperbola is based.[112]

§. 77. Afterward, if for the hyperbola we want to express ξ, η by means of u, we shall easily found that, by using the infinite series, their form must be[113]

$$\xi = 1 + Au^2 + Bu^4 + Cu^6 + \&c.$$
$$\eta = au + bu^3 + cu^5 + du^7 + \&c.$$

Since, by doing $u = 0$, we have $\xi = 1$, $\eta = 0$. In addition, by taking u infinitely small, ξ will increase as much as u^2, & η will increase as much as u, since the angle at A is right, & the osculating radius of the hyperbola in A is =AC. Finally, by taking u to be negative, all the values of ξ will be the same as for u being positive, from which it follows that the abscissa ξ must be expressed by the even dimensions of u. And by taking u to be negative, the values of η will be the same ones, but negative. Therefore η must be expressed by the odd dimensions of u. Therefore, all that remains is to determine the coefficients. This is what the two formulas found above will serve us for

$$d\xi : du = \eta,$$

$$d\eta : du = \xi.$$

We shall therefore have, by differentiating the first series

$$d\xi : du = 2\mathrm{A}u + 4\mathrm{B}u^3 + 6\mathrm{C}u^5 + \cdots\cdots + \mu \cdot \mathrm{M}u^{\mu-1}$$

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[109] Taking into account the previous relationship, the equality between Mmp and Nnq seems to be based on the similarity with CMP and CNQ respectively, which share the angle φ.

[110] There is an error here. It should be: «Vmp + Mmp = 90°, and so Vmp = Cmp».

[111] There is an error here. It should be: «just like in the circle the normal on n is equal to Cn ».

[112] He goes on to address the central part of the issue (§. 77. y §. 79.).

[113] Let us think $\xi \equiv \xi(u)$ and $\eta \equiv \eta(u)$ as functions of u. Although he does not mention it, Lambert is likely to be drawing on Taylor's expansion around 0 of both $\xi(u)$ and $\eta(u)$ («... taking u infinitely small ...» he says in the following lines, and explains that «... doing $u = 0$, we have $\xi = 1, \eta = 0$ »). The reason why only the even powers appear in the first expression and the odd powers in the second one, is that the first function is even and the second odd, since the abscissa does not vary if the argument is negative (this means that the area falls below the abscissa axis), but the ordinate changes its sign, that is:

$$\xi(-u) = \xi(u) \quad \textit{and} \quad \eta(-u) = -\eta(u)$$

that must be $= \eta$, hence

$$d\xi : du = au + bu^3 + cu^5 + \cdots\cdots + m \cdot u^{\mu-1}$$

Therefore by comparing terms

$$\begin{aligned} 2\mathrm{A} &= a, \\ 4\mathrm{B} &= b, \\ 6\mathrm{C} &= c, \\ &\&c. \\ \mu\mathrm{M} &= m. \end{aligned}$$

But, differentiating η, it must also be $d\eta : du = \xi$, hence

$$\begin{aligned} d\eta : du &= a + 3bu^2 + 5cu^5 + \cdots\cdots (\mu - 1) \cdot mu^{\mu-2} \\ &= 1 + \mathrm{A}u^2 + \mathrm{B}u^4 + \cdots\cdots\cdots \mathrm{L} \cdot u^{\mu-2} \end{aligned}$$

Therefore by comparing terms

$$\begin{aligned} q &= 1, \\ 3b &= \mathrm{A}, \\ 5c &= \mathrm{B}, \\ &\text{\&c.} \\ (\mu - 1)m &= \mathrm{L}. \end{aligned}$$

By means of these equations we shall have

$$\begin{aligned} a &= 1, \\ \mathrm{A} &= \frac{1}{2}a = \frac{1}{2}, \\ b &= \frac{1}{3}\mathrm{A} = \frac{1}{2 \cdot 3}, \\ \mathrm{B} &= \frac{1}{4}b = \frac{1}{2 \cdot 3 \cdot 4}, \\ c &= \frac{1}{5}\mathrm{B} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}, \\ \mathrm{C} &= \frac{1}{6}c = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}, \end{aligned}$$

&c.

$$m = \frac{1}{(\mu-1)}\mathrm{L} = \frac{1}{2\cdot3\cdot4\cdot\cdots\cdot(\mu-1)},$$

$$\mathrm{M} = \frac{1}{\mu}\cdot m = \frac{1}{2\cdot3\cdot4\cdot\cdots\cdots\mu}.$$

Thus we shall have

$$\xi = 1 + \frac{1}{2}u^2 + \frac{1}{2\cdot3\cdot4}u^4 + \frac{1}{2\cdot3\cdot4\cdot5\cdot6}u^6 + \text{\&c.}$$

$$\eta = u + \frac{1}{2\cdot3}u^3 + \frac{1}{2\cdot3\cdot4\cdot5}u^5 + \frac{1}{2\cdot3\cdot4\cdot5\cdot6\cdot7}u^7 + \text{\&c.}$$

Hence here is the abscissa ξ, & the ordinate η expressed by the letter u, which is twice the area of the hyperbolic segment AMCA. Now we known that if instead of u, we take v, which is twice the circular segment ANCA, the abscissa x, & the ordinate y, both circular, are[114]

$$x = 1 - \frac{1}{2}v^2 + \frac{1}{2\cdot3\cdot4}v^4 - \frac{1}{2\cdot3\cdot4\cdot5\cdot6}v^6 + \text{\&c.}$$

$$y = v - \frac{1}{2\cdot3}v^3 + \frac{1}{2\cdot3\cdot4\cdot5}v^5 - \frac{1}{2\cdot3\cdot4\cdot5\cdot6\cdot7}v^6 + \text{\&c.}$$

two series which in form do not differ from the two preceding ones except by the alternative change of signs.

§. 78. And since we have (§. 73.)

$$\frac{e^u + e^{-u}}{2} = 1 + \frac{1}{2}u^2 + \frac{1}{2\cdot3\cdot4}u^4 + \text{\&c.}$$

$$\frac{e^u - e^{-u}}{2} = u + \frac{1}{2\cdot3}u^3 + \frac{1}{2\cdot3\cdot4\cdot5}u^5 + \text{\&c.}$$

we see we shall have

$$\xi = \frac{e^u + e^{-u}}{2},$$

114 There is an error in the second formula. It should be written as follows:

$$x = 1 - \frac{1}{2}v^2 + \frac{1}{2\cdot3\cdot4}v^4 - \frac{1}{2\cdot3\cdot4\cdot5\cdot6}v^6 + \text{\&c.}$$

$$y = v - \frac{1}{2\cdot3}v^3 + \frac{1}{2\cdot3\cdot4\cdot5}v^5 - \frac{1}{2\cdot3\cdot4\cdot5\cdot6\cdot7}v^7 + \text{\&c.}$$

$$\eta = \frac{e^u - e^{-u}}{2},$$

& that consequently these quantities express the abscissa $\xi =$ CP, & the ordinate $\eta =$ PM of the hyperbola.

§. 79.[115] And since $\eta : \xi = \operatorname{tang}\varphi$, we also see that

$$\operatorname{tang}\varphi = \frac{e^u - e^{-u}}{e^u + e^{-u}},$$

therefore by §. 81.[116]

$$\operatorname{tang}\varphi = \cfrac{1}{1:u + \cfrac{1}{3:u + \cfrac{1}{5:u + \cfrac{1}{7:u + \cfrac{1}{9:u + 1\ \text{\&c.}}}}}}$$

And since the same tangent is also

$$\operatorname{tang} v = \operatorname{tang}\varphi = \cfrac{1}{1:v - \cfrac{1}{3:v - \cfrac{1}{5:v - \cfrac{1}{7:v - \cfrac{1}{9:v - 1\ \text{\&c.}}}}}}$$

we see that this tangent is found by these two continued fractions, which differ in form only in their signs: it is only a matter of using $u =$ 2AMCA when the first one is used, instead of using $v =$ 2ANCA in order to have the same tangent by means of the second one. Hence here it is the analogy that had to be found independently of, & without incorporating, the imaginary quantities.

§. 80. Now we can draw in very clear terms the consequence that *the area of the hyperbolic sector* AMCA, *as well as that of the corresponding circular sector* ANCA, *will be an irrational or incommensurable quantity to the square of the radius* AC, *as long as the angle* φ, *which is the one formed by one & the other of the two*

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115 A new note to this analogy: $\tanh u = \tan v$.

116 This call to a later point is strange, considering also that it is something that appears in the last part of §. 73 (the only difference is the argument of the functions).

sectors with the center C, *has a rational tangent*, & that conversely *this tangent will be irrational as long as one of the two sectors is a rational quantity*.[117]

§. 81. There is an absolutely similar consequence to be drawn in relation to the continued fraction[118] (§. 74.)

$$\frac{e^u+1}{2} = \cfrac{1}{1 - \cfrac{1}{2:u + \cfrac{1}{6:u + \cfrac{1}{10:u + \cfrac{1}{14:u + \cfrac{1}{18:u + \&c.}}}}}}$$

which transforms into

$$\frac{e^u+1}{2} = \cfrac{1}{1 + \cfrac{1}{-2:u + \cfrac{1}{-6:u + \cfrac{1}{-10:u + \&c.}}}}$$

& from where we draw for u negative

$$\frac{e^u+1}{2} = \cfrac{1}{1 + \cfrac{1}{2:u + \cfrac{1}{6:u + \cfrac{1}{10:u + \cfrac{1}{14:u + \&c.}}}}}$$

[117] New irrationality results based on the fact that $\tanh u = \tan v$. It follows from this, on the one hand, that:

$$\text{If } \tan\varphi \in \mathbb{Q} \Rightarrow u,\ v \notin \mathbb{Q},$$

and, on the other hand, that a sufficient condition for the irrationality of the hyperbolic tangent:

$$\text{If } u \text{ or } v \in \mathbb{Q} \Rightarrow \tan v = \tanh u \notin \mathbb{Q}$$

[118] This consequence is —as it is written at the end of this point— a new result of irrationality, in this case for natural logarithms:

$$\text{If } u \in \mathbb{Q} \Rightarrow \ln u \notin \mathbb{Q}$$

These fractions let us know *to what extent the irrationality of the number* $e = 2,718281828459045235360 28 \cdots$ *is transcendental, inasmuch as none of its powers or roots are rational.*[119] Since u & e^u cannot be a rational quantity at the same time. Now, since u is the hyperbolic logarithm of e^u, it follows that *every rational hyperbolic logarithm is that of an irrational number*, & that conversely, *every rational number has an irrational hyperbolic logarithm.*

§. 82. But now let us see what e^u & e^{-u} mean in the figure. Let us return for this effect to §. 78. where we found the two formulas

$$\xi = \frac{e^u + e^{-u}}{2},$$

$$\eta = \frac{e^u - e^{-u}}{2},$$

hence by taking the sum & the difference, we shall have

$$e^u = \xi + \eta,$$

$$e^{-u} = \xi - \eta.$$

But the asymptotes CF, CS, forming a right angle between them, which the axis CH cuts into two equal parts, will give

$$\xi = \text{CP} = \text{PS} = \text{PR},$$
$$\eta = \text{PM},$$

hence

$$\xi + \eta = \text{SM},$$

$$\xi - \eta = \text{MR},$$

& therefore

$$e^u = \text{SM},$$

$$e^{-u} = \text{MR},$$

---

119 Again, as he did at the beginning of this *Mémoire* at point §. 2 (and in fact in more places), Lambert again uses an expression of the style «to what extent the irrationality of a certain quantity is transcendental». Contrary to the already mentioned cases, he does provide here an idea of what this means: «*inasmuch as none of its powers or roots are rational*», which in particular means that $e^n \neq q$, with q rational, that is, e is not the root of a wide variety of algebraic equations. Implicit in this sentence —let us apply a «proof by simplicity» in the style of the one he himself uses at the beginning of this article— we therefore find a transcendence conjecture for e.

from where we see at the same time that we shall have

$$e^u \cdot e^{-u} = \mathrm{SM} \cdot \mathrm{MR} = 1$$

We see, moreover, that while we have

$$e^u = \mathrm{SM},$$

$$e^{-u} = \mathrm{MR},$$

$$\mathrm{AB} = 1,$$

by taking logarithms we shall have

$$u = \log \frac{\mathrm{SM}}{\mathrm{AB}} = \log \frac{\mathrm{AB}}{\mathrm{MR}}.$$

And since u, e^u, could not be rational at the same time, we see the same thing in relation to the area of the sector AMCA= $\frac{1}{2}u$, & the ordinates SM, MR.

§. 83.[120] We also have (§. 75.) the differential

$$du = \frac{d\,\mathrm{tang}\varphi}{1 - t\varphi^2},$$

whose integral turns out to be[121]

$$2u = \log \frac{1 + t\varphi}{1 - t\varphi} = \log .\mathrm{tang}.(45^\circ + \varphi) = \mathrm{l.tang\,SCM},$$

or

$$2u = -\log \frac{1 - t\varphi}{1 + t\varphi} = -\mathrm{l.tang}(45^\circ - \varphi) = -\mathrm{l.tang\,RCM}.$$

Let us take the first of these formulas

$$2u = \log \left(\frac{1 + t\varphi}{1 - t\varphi} \right),$$

& it will put us in the situation of finding also in relation to the hyperbolic sectors, that which we have seen to be the *prime tangent* in relation to the circular sectors. Here it is how.

[120] What he is going to do now (§. 83–§. 87) is to show how the concept of *prime tangent* applies equally to the hyperbolic case. The point §. 87 enunciates all the derived theorems but translated to the hyperbolic case, which shows a new connection between the c.t.f. and the h.t.f.

[121] Lambert will be using both «log» and «l.» to denote the logarithm.

§. 84. Let us first consider that the hyperbolic sector AMCA increases with the angle φ =MCA, in such a way that it becomes infinite, when $\varphi = 45°$. It is therefore clear that given one of these sectors, others can be found which are any multiples of it, & any parts of it, or exceed it by any quantity. Now, to each of these sectors corresponds an angle MCP by means of which it is formed, & the tangent of this angle being $= \varphi$, and the sector $= \frac{1}{2}u$, we just saw that

$$2u = \log \frac{1 + t\varphi}{1 - t\varphi}.$$

§. 85. Let now be three sectors $\frac{1}{2}u,\ \frac{1}{2}u',\ \frac{1}{2}u''$, such that the third is the sum of the first two. Also let $\varphi,\ \varphi',\ \varphi''$ be the corresponding angles. And we shall have

$$2u = \log \frac{1 + t\varphi}{1 - t\varphi},$$

$$2u' = \log \frac{1 + t\varphi'}{1 - t\varphi'},$$

$$2u'' = \log \frac{1 + t\varphi''}{1 - t\varphi''}.$$

Therefore it must be

$$\frac{1}{2}u'' = \frac{1}{2}u' + \frac{1}{2}u,$$

and equally

$$\log \frac{1 + t\varphi''}{1 - t\varphi''} = \log \frac{1 + t\varphi'}{1 - t\varphi'} + \log \frac{1 + t\varphi}{1 - t\varphi},$$

which gives

$$\frac{1 + t\varphi''}{1 - t\varphi''} = \frac{1 + t\varphi'}{1 - t\varphi'} \cdot \frac{1 + t\varphi}{1 - t\varphi},$$

from where it follows

$$t\varphi'' = \frac{t\varphi + t\varphi'}{1 + t\varphi \cdot t\varphi'},$$

& reciprocally for the difference

$$t\varphi' = \frac{t\varphi'' - t\varphi}{1 - t\varphi \cdot t\varphi''}.$$

These two formulas do not differ more than in the signs in relation to those that we have found for the sectors, or the circular arcs, & allow us equally to conclude that *if the tangents corresponding to two hyperbolic sectors are rational, the tangents* *corresponding to the sector equal to the sum & the difference of these two sectors will be equally rational.*

§. 86. This proposition alone is enough to show that everything we have said above (§. 52 ... 71.) in relation to the circle, will apply equally to the hyperbola. We only have to use an abbreviated way of speaking, calling *tangent of any hyperbolic sector* ACMA, the tangent of the angle ACM, which is =AT, the radius being AC = 1. Afterward, it should be noted that all the sectors discussed here must have the AC axis as a common origin, as the sectors MCAM, *m*CA*m* do. Thus, for example, the sector *m*CM not touching the axis, must be replaced by another one that is equal, & that is contiguous to the axis AC, if we want to have the angle φ and the corresponding tangent. It is clear that this observation is not necessary when it comes to the circle, since any diameter of the circle can be considered as an axis.

§. 87. It is therefore in this sense that I will say that *the hyperbola has an infinity of prime tangents*, that *the sectors of all these prime tangents are incommensurable with each other & to unity*, that *the tangent of a sector being prime, there are no more rational tangents than those of the multiples of that sector*: That *every rational tangent is either itself prime, or its sector is a multiple of a sector whose tangent is prime.* &c. Since the proof of these theorems would be but a repetition of those I have given for the circle, I will omit them all the more since I have not reported these theorems but to show again at this point the analogy between the circle & the equilateral hyperbola.

§. 88.[122] Let us compare again the circular sector ANCA, & the hyperbolic sector AMCA. Mr. *de Foncenex*, in the Memoire cited above (§. 74.) has shown that by using imaginary quantities, these two sectors turn out to be in the ratio 1 to $\sqrt{-1}$, which is purely imaginary. Now, what will be the real reason? This is what we shall find by expressing one of these sectors through the other. For this purpose we shall use the two series[123]

[122] In Barnett (2004, p. 24) it can be read:

> Foncenex himself went no further in exploring "this affinity" than to conclude that, since $\sqrt{x^2 - r^2} = \sqrt{-1}\sqrt{r^2 - x^2}$, "the circular sectors and hyperbolic [sectors] that correspond to the same abscissa are always in the ratio of 1 to $\sqrt{-1}$". It is this use of an imaginary ratio to pass from the circle to the hyperbola Lambert seemed intent on avoiding.

This is what this point is about. In any case, it is not a petty thing to make a nuance. No doubt Lambert lives in a specific historical era, and in this way we must understand this type of elusiveness in the use of complex quantities. But as an individual with a concrete personality and thought, and indeed with an open mind, Lambert was a person «without any fear of the imaginary» (Engel et al. 1895, pp. 145–146) unlike his contemporaries, as corroborated by his conjecture on the sphere of imaginary radius (see the correspondent appendix). An example of such an internal struggle between their absurd character —as far as their existence is concerned— and their usefulness, which can be found in several authors following the first appearance of imaginary quantities, is reflected in a letter to Kant: «The sign $\sqrt{-1}$ represents a non-entity that is not thinkable, and yet it can very well be used to find theorems» (the reference of F. Engel and the letter to Kant, by José Ferreirós).

[123] There is an error here. It should be:

$$v = t\varphi - \frac{1}{3}t\varphi^3 + \frac{1}{5}t\varphi^5 - \frac{1}{7}t\varphi^7 + \&c.$$
$$t\varphi = v - \frac{1}{3}u^3 + \frac{2}{15}u^5 - \frac{17}{315}u^7 + \&c.$$

which are easily found by the differential formulas given above (§. 75.). By substituting hence the value of the second of these series in the first one, we shall have, after reducing,

$$v = u - \frac{2}{3}u^3 + \frac{2}{3}u^5 - \frac{244}{315}u^7 + \&c.$$

& reciprocally

$$u = v + \frac{2}{3}v^3 + \frac{2}{3}v^5 + \frac{244}{315}v^7 + \&c.$$

These two series only differ in relation to signs, having the same coefficients & exponents. If in the first of these series we put

$$u = v\sqrt{-1},$$

we obtain[124]

$$v = \sqrt{-1} \cdot \left(v + \frac{2}{3}v^3 + \frac{2}{3}v^5 + \frac{244}{315}v^7 + \&c.\right)$$

which means

$$v = u\sqrt{-1}.$$

Therefore, by means of an imaginary hyperbolic sector, we obtain an imaginary circular sector, & reciprocally.

§. 89. All that I have just shown about the circular & logarithmic transcendental quantities seems to be based on much more universal principles, but which are not yet sufficiently developed. However, here is what may serve to give some idea. It is not enough to have found that these transcendental quantities are irrational, that is, incommensurable to unity.[125] This property is not unique to them.[126] Since, in addition to the fact that there are irrational quantities that can be formed at random, &

$$v = t\varphi - \frac{1}{3}t\varphi^3 + \frac{1}{5}t\varphi^5 - \frac{1}{7}t\varphi^7 + \&c.$$
$$t\varphi = u - \frac{1}{3}u^3 + \frac{2}{15}u^5 - \frac{17}{315}u^7 + \&c.$$

[124] See the note by A.S. in Appendix C.

[125] The first thing to note is that the term «transcendental» does not yet have the modern meaning here, since it would not be necessary —from the modern point of view— to demonstrate that a transcendental quantity is irrational, as it is by definition.

[126] The second key point is to understand appropriately this phrase. What Lambert is saying is that there are certain quantities —the «transcendental» quantities— which by their special nature are

which are thus hardly within the competence of analysis, there are still an infinity of others that we call *algebraic*:[127] & such are all the *radical irrational quantities*, like $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{4}$ & c. $\sqrt{\left(2+\sqrt{3}\right)}$ & c. & all the *irrational roots of algebraic equations*, such as, for example, those of the equations

$$\begin{aligned} 0 &= xx - 4x + 1, \\ 0 &= x^3 - 5x + 1, \\ &\&c. \end{aligned}$$

I will call the ones & the others *radical irrational quantities*, & here is the theorem which I think can be proved.[128]

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not sufficiently well represented under the label «irrational», which leads him to make a distinction between irrationals. But: how does he make this distinction?

[127] Here we have how Lambert makes this distinction:

quantities «formed at random» vs. algebraic quantities

However, from Lambert's perspective —as he makes it clear in the following lines—, algebraic quantities are no longer those that can be expressed through a finite combination of algebraic operations, following the Leibnizian and Eulerian tradition, but quantities that are roots of algebraic equations. Therefore, those quantities «formed at random», «transcendental» in the classical sense of the term —that is to say, non-expressible quantities—, are the quantities that are not roots of algebraic equations. Consequently, the term transcendental here acquires its modern meaning and therefore Lambert's distinction between irrationals is the modern one:

transcendental quantities vs. algebraic quantities

[128] Two comments are worth making here. The first is a call to be cautious, for despite the enthusiasm one may feel when witnessing Lambert's distinction between algebraic and transcendental quantities in the modern sense of the terms, everything seems to indicate that although he explicitly identifies algebraics with roots, he implicitly continues to identify algebraics with radicals, that is, with the classical idea of «expressible quantities». What happens is that he uses as a yardstick to make the classification among irrationals the idea of «being root» and not that of «being expressible», since it is simpler and more manageable (this idea is also expressed in Serfati (2018, pp. 182–183), although the interpretation I give in this note seems to differ slightly from the one provided there). The clearest call for attention in this direction is the change in the name that he gives to algebraics: «I will call the ones & the others *radical irrational quantities...*», precisely the name he gives to those quantities that are explicitly radicals. Another call for attention is historical, being very likely that Lambert was assuming the —for us— old idea that every equation is solvable by radicals, a general feeling at the time until the end of the 18th century, and that therefore he was identifying «root» with «expressible by radicals» Sorensen (2010, p. 2) and specifically Sorensen (2010, Chap. 4: pp. 66, 29–32) (this is also mentioned in Ayoub (1980, p. 262) where it is said that apparently nobody suspected that, in particular, the equation of degree 5 could not be solved. Furthermore, he comments on the basis of the (vague) approval of both the Royal Society and Cauchy, and Lagrange's reaction, that the mathematical community was not prepared to accept such a result, hence the general non-acceptance of Ruffini's work Ayoub (1980, pp. 271–272)). The second comment is that independently of this, this theorem, «which I think can be proved», is a clear conjecture of transcendence in the modern sense of the term.

§. 90. I hence say *that no circular & logarithmic transcendental quantity can be expressed through some radical irrational quantity, which is related to the same unit, & into which no transcendental quantity enters*.[129] This theorem seems to must be proved from the fact that transcendental quantities depend on

$$e^x,$$

where the exponent is variable, while radical quantities suppose constant exponents. Thus e.g.[130] an arc of a circle being rational or commensurable to the radius, the tangent, which we have seen to be irrational, cannot be a square root of a rational quantity. Since if the proposed arc is $= \omega$, & we do $\operatorname{tang} \omega = \sqrt{a}$, we shall have

$$t\omega^2 = \frac{\int \omega^2}{\cos \omega^2} = \frac{1 - \cos 2\omega}{1 + \cos 2\omega} = a,$$

from where it follows

$$\cos 2\omega = \frac{1-a}{1+a},$$

but this quantity being rational, it follows that the arc 2ω is irrational, which is contrary to the hypothesis, it is clear that by doing $\operatorname{tang} \omega = \sqrt{a}$, the quantity a cannot be rational, & that therefore the tangent of any rational arc is not a square root of a rational quantity.

§. 91. Once this theorem is proven in all its universality, it will follow that the circumference of the circle cannot be expressed by some radical quantity, nor by some rational quantity, there will be no way to determine it by some geometric construction. Since everything that can be constructed geometrically corresponds to rational & radical quantities;[131] & it is far from being the case that the latter can be

[129] Based on what has been pointed out in the previous notes, this is a conjecture of transcendence for the type of quantities that he calls «circular» and «logarithmic», among which are e and π. In case more clarification was needed, Lambert writes in a letter to Holland:

> The way I have proved this [the irrationality of π] can be extended to the point that circular and logarithmic quantities cannot be roots of rational equations

(quoted in Cantor IV (1908, p. 447, note 6)).

[130] In order to support his conjecture, he proves the following particular case: $\tan v \neq \sqrt{q}$, with v and q rationals. The conclusion is that $\tan v$ (with v a rational quantity) is not a quadratic irrational.

[131] Here Lambert connects the (conjectured) transcendence of π with the impossibility of squaring the circle: the circle cannot be squared because it would require that π be geometrically constructed, which is not because it is transcendental. Serfati (2018, p. 183) sums up how correct and pioneering this reasoning is:

> The procedure described here is precisely the one that Wantzel, then Hermite, and then Lindemann will offer [...] Lambert appears here as an irreproachable visionary.

indifferently constructed.[132] It is clear that it will be the same for all the arcs of circles where the length or the two extreme points are given, either by rational quantities or by radical quantities. Since, if the length of the arc is given, we shall have to find its two extreme points by using the chord, the sine, the tangent, or any other straight line that, in order to be constructed, will always be dependent on or reducible to one of the lines that I have just named. But if the length of the arc is given by rational or radical quantities, these lines will be transcendental, and thus irreducible to some rational or radical quantity. It will be the same if the extreme points of the arc are given, by which I mean by rational or radical quantities. Since, in this case, the length of the arc will be a transcendental quantity: which means irreducible to some rational or radical quantity, & therefore it does not admit any geometric construction.[133]

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[132] That is, everything that is constructible is algebraic, but not the other way around (Serfati 2018, p. 183).

[133] At the very end of this *Mémoire*, Lambert makes it clear that the term «transcendental» denotes quantities that are not roots of equations, and that being transcendental implies not being constructible.

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Appendix A
About Lambert's Portrait

Lambert, a man «with a very particular physiognomy», never wanted to have his portrait painted.[1] It seems that Johann III Bernoulli, one of his best friends, published a caricature in 1786 «saying it was a good likeness».[2] Said caricature was likely the base for the whole-body portrait —left part of the image[3]— designed by the German painter and director of the Berlin Academy of Arts Daniel Chodowiecki, a portrait that would eventually be engraved in Berlin in 1812 by Daniel Berger. The artist Pierre Roch Vigneron expanded the upper part to elaborate a design in commemoration of the centenary of the birth of Lambert in 1828, a design that would finaly pass to be lithographed by the Mulhousien G. Engelmann.[4]

[1] Jaquel (1969, p. 302).

[2] Gray (2007, p. 84 note 5).

[3] Available at http://ark.bnf.fr/ark:/12148/cb41920300d.

[4] See Jaquel (1969, p. 302) and Jaquel (1967/68) (I want to express again my gratitude to Eliane Michelon from *Archives de Mulhouse* for having facilitated me this last work).

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This image appeared for the first time in a publication in 1828 in the minutes of the ceremony held in Mulhouse on the occassion of the centenary of the birth of Lambert.[5] The image also appeared in 1829 in Huber et al. (1829), «a volume that will be the authorized work about Lambert throughout the XIX century».[6] It has, since that time, been the image generally used to represent Lambert. On the lower part of that lithograph, some handwritten lines by Lambert in facsimile edition expressing «vigorously, even with elegance, the teleological convictions of the savant» were included:[7]

> Organic bodies, amongst all the others, are the most abundantly and easily originated on our earth... Everything for which the means are most abundantly available in the world must be considered as part of the purposes of Creation.

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[5] *Gedächtnissfeier von Johann Heinrich Lambert begangen in Mühlhausen den 27ten August 1828*, Beschrieben durch Franz Christian Joseph, evangelischen Pfarrer zu Mühlhausen und Sekretär des Lambert'schen Vereins, 1828 (see Jaquel (1967/68)).

[6] Jaquel (1977, p. 5).

[7] Jaquel (1967/68).

Appendix B
Lambert and Non-Euclidean Geometry

Although mathematics at this time was centered around analysis, certain questions had never ceased to worry and intrigue mathematicians. From the beginning they had questioned the lack of evidence of the fifth postulate of the *Elements* in comparison with the others, along with the fact that its converse did require explicit demonstration. The point, it should be noted, was not whether this statement was true or false; the force of intuition indicated the answer. It was rather whether this statement was a fundamental truth or not. This issue ended up becoming a thorn in geometry's side for a long time, Lambert marking the path that those who settled the question almost a century later would follow.

The work that marked the turning point in all this was *Euclides ab omni naevo vindicatus* (1733) by Giovanni Girolamo Saccheri.[8] In this work, Saccheri started from a quadrilateral with two right angles ($A = B = 90°$) and two equal sides ($AC = BD$), today known as Saccheri's quadrilateral,[9] and demonstrated based only on the first 28 propositions (those that do not use the axiom of parallels) that in this case the angles C and D must be the same. There are therefore three options: they are right angles —which is equivalent to the fifth postulate being satisfied—, acute angles, or obtuse angles. This generates three types of geometries,[10] which, as he also showed, are mutually exclusive: the right angle geometry (RAG) which is equivalent to the Euclidean, the acute angle geometry (AAG) and the obtuse angle geometry (OAG). The method followed by Saccheri is his most valuable and profound contribution:[11] adding to the first four postulates the negation of the fifth and trying to establish that this necessarily leads to a contradiction; that is to say, demonstrating that the OAG and AAG are not possible.

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[8] *Euclid Vindicated from Every Blemish*. For an English annotated translation see Saccheri (1733).

[9] See Fig. B.1.

[10] Each of the previous options, if it is true for a Saccheri's quadrilateral, it is true for all (which he proves), which justifies our speaking of three types of geometries or, following his terminology, three different hypotheses (RAH, AAH and OAH).

[11] See Dou (1992, p. 52).

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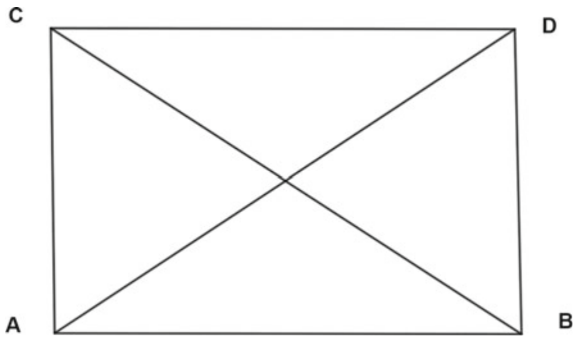

Fig. B.1 Saccheri's quadrilateral

Today we know that such geometries are possible, but with nuances. The OAG used by Saccheri and that he correctly refuted, presupposes the infinity of straight lines (second postulate). In this respect it does not fully concord with elliptical geometry, in which these lines are of finite length. Furthermore, straight lines in the elliptical plane differ so much from what a straight line apparently is (they are of finite length, not uniquely determined by two points), that OAG was discarded as a possible geometry. Saccheri commited the error in the case of AAG, the so-called hyperbolic geometry:

> Saccheri did not find the supposed contradiction, as it was nowhere to be found, but he was unable to convince himself that the new geometry he had erected might in fact be a reasonable alternative to Euclid's Elements rather than a green-eyed monster.[12]

Although Saccheri's work is mentioned in some scientific journals of the epoch as *Acta Eruditorum*, as well as in the first books on the history of mathematics by J. C. Heilbronner (Leipzig, 1742) and J. E. Montucla (Paris, 1758), it went almost unnoticed for almost two centuries, with the notable exception of German-speaking territories, where his study would be particularly intense.[13] In fact, it was here that the interest on this topic arose with more intensity. Küstner came to assemble a large library on the problem of the parallels, and Klügel wrote a thesis under his direction (1763) in which he studied exhaustively the different attempts to demonstrate the fifth postulate, in particular Saccheri's. The conclusion he reached is that although these results may contradict experience, they do not contradict the axioms. Klügel seems to have been the first to recognize the possibility of the fifth postulate's being independent.[14]

[12] Saccheri (1733, p. 4).

[13] See Saccheri (1733, pp. 49–58).

[14] See Ewald (1996, p. 155).

Therefore, Lambert arrived in Güttingen «immersed» in the problem of parallels. When Klügel's thesis reached his hands after a few years, his interest in the subject is materialized in the work *Theorie der Parallellinien* (1766/1786).[15] In the third part, which is the one containing the most important contributions, he introduced Lambert's quadrilateral, which differs from Saccheri's quadrilateral in that it has three right angles.[16] The remaining angle gave rise to the same three hypotheses or geometries discussed earlier: the RAG equivalent to the Euclidean, the OAG that was discarded after arriving at a contradiction with the other postulates,[17] and AAG. Continuing in this direction, Lambert realized the difficulties involved in finding a contradiction and proved a series of results that, compared with those of the OAG, gave rise to fertile observations:

First, Lambert established two results for triangles that are satisfied in the case of AAG, namely:

$$A + B + C < \pi$$

$$\mathcal{A}_t = k(\pi - A - B - C),$$

being completely analogous to those of the OAG:

$$A + B + C > \pi$$

$$\mathcal{A}_t = k(A + B + C - \pi),$$

(A, B and C being the angles, and k the constant of proportionality).

After that, he established the connection between a hypothetical geometry (OAG) and a «real» geometry (that of the sphere), allowing him to jump into a conjecture about the AAG, since in a sphere the sum of the angles of a triangle is greater than

[15] *Theory of parallel lines* Lambert (1766/1786). Lambert wrote this work in 1766, but it was not published until 1786. The reader will find a modern French annotated translation in Papadopoulos and Théret (2014b). An outline of this work by Lambert is also summarized by the same authors in Papadopoulos and Théret (2014a), and can be also found in Bonola's book Bonola (1912, pp. 45–50) (I am grateful to one of the anonymous reviewers for suggesting to include here a reference to Bonola's classic book). As to why Lambert did not publish this work in life, see Dou (1970, pp. 400, 401, 411 note 38).

[16] On a possible connection between Lambert and Saccheri, one can read in Saccheri (1733, p. 53) that:

> It is unclear whether Lambert had first-hand acquaintance with Saccheri's book, but this hypothesis is not really necessary to explain his achievement, given Lambert's genius and the abundance of details provided by Klügel's dissertation.

[17] He proved that two different straight lines share two different points. This again does not contradict the geometry of the sphere where the lines are of finite and constant length and share two points (the poles), but does contradict the first four postulates of Euclid and therefore the OAG.

π and its area is also proportional to its excess. Specifically, R being the radius of the sphere, we have that:

$$\mathcal{A}_t = R^2(A + B + C - \pi)$$

If we now look at the formula for the area of a triangle under the AAH, and that of one triangle on the spherical surface, we see that the difference is minimal. In fact, if we make a formal substitution in the latter, R by Ri, where i is the imaginary unit, then we obtain the first one:

$$\begin{aligned}\mathcal{A}_t &= (Ri)^2\,(A + B + C - \pi)\\ &= -R^2(A + B + C - \pi)\\ &= R^2(\pi - A - B - C)\\ &= k(\pi - A - B - C)\end{aligned}$$

Lambert did not make it explicit, but it is very possible that this was what led him to his conjecture: «I am tempted to conclude that the third hypothesis holds for some imaginary sphere».[18]

These ideas were tremendously fruitful, paving the way for the development of hyperbolic geometry, a path that Beltrami would culminate by putting this geometry at the same level of consistency as the Euclidean.[19] As for Lambert, he was not capable of finding the long-awaited contradiction. In this respect, Papadopoulos and Théret (2014a, p. 21) concludes that:

> The text is unfinished, and it is not possible to know for sure whether Lambert was convinced whether the parallel postulate is a theorem or not.

[18] See Abardia et al. (2012, p. 294 note 6).

[19] For an in-depth study of the impact of Lambert's work on non-Euclidean geometry, see *Una revisión de la historia del descubrimiento de las geometrías no euclídeas*, a doctoral thesis which included Abardia et al. (2012) and Rodríguez (2006).

Appendix C
Notes by Andreas Speiser

In this appendix we provide the translation of the notes that Andrea Speiser included in his edition of the *Mémoire*. Some brief remarks about some of these notes will also be included when necessary.

Notes to §. 7. Speiser (1946–1948, p. 115).

(1) Original: even.[20] A.S.
(2) Original: odd. A.S.

Notes to §. 8. Speiser (1946–1948, p. 116).

(1) The formulas should be written as follows:[21]

$$R^{4n+1} = -\frac{2^{4n+1}(1 \cdot 2 \cdots (4n+1))}{(8n+3)!} v^{4n+2} + \frac{2^{4n+1}(2 \cdot 3 \cdots (4n+2))}{(8n+5)!} v^{4n+4} - \text{etc.}$$

$$R^{4n+2} = -\frac{2^{4n+2}(1 \cdot 2 \cdots (4n+2))}{(8n+5)!} v^{4n+3} + \frac{2^{4n+2}(2 \cdot 3 \cdots (4n+3))}{(8n+7)!} v^{4n+5} - \text{etc.}$$

$$R^{4n+3} = +\frac{2^{4n+3}(1 \cdot 2 \cdots (4n+3))}{(8n+7)!} v^{4n+4} - \frac{2^{4n+3}(2 \cdot 3 \cdots (4n+4))}{(8n+9)!} v^{4n+6} + \text{etc.}$$

A.S.

Notes to §. 11. Speiser (1946–1948, p. 117).

[20] Speiser made the correction in the text. In other occassions, he kept the mistake in the text including the correction in a footnote.

[21] These expressions are equivalent to those offered in our edition.

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(1) Based upon the corrections pointed out in the previous note.[22] A.S.

Notes to §. 19. Speiser (1946–1948, p. 122).

(1) Original edition here and in the following formulas:[23] $a^n = \dfrac{1}{(2n+1)w - a^{n+1}}$.
Corrected by A.S.

Notes to §. 34. Speiser (1946–1948, p. 131).

(1) Original edition:[24]

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540} + \frac{1}{540 \cdot 5879} + $$
$$+ \frac{1}{5879 \cdot 75587} + \frac{1}{75587 \cdot 1147426} + \text{etc.}$$

Corrected by A.S.

Notes to §. 55. Speiser (1946–1948, p. 139).

(1) Original edition: rational. Corrected by A.S.

Notes to §. 72. Speiser (1946–1948, p. 144).

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[22] The correction made by Speiser in the text is:

$$\pm r^n = -\frac{2^n \cdot m \cdot (m+1) \cdot (m+2) \cdot \cdots \cdot (n+m-1) v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+2m-1)}$$

$$\pm r^{n+1} = -\frac{2^{n+1} \cdot m \cdot (m+1)(m+2) \cdots (n+m) v^{n+2m}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+2m+1)}$$

$$\pm r^{n+2} = -\frac{2^{n+2} \cdot (m-1) \cdot m \cdot (m+1) \cdots (n+m) \cdot v^{n+2m-1}}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+2m+1)}$$

[23] The correction made by Speiser is:

$$a^n = \frac{1}{(2n+3)w - a^{n+1}}$$

Note that Speiser slightly changed the expression keeping the upper indexes, while in this edition we keep the expression and change the upper indexes.

[24] There is a mistake in the text that had gone unnoticed by me. Speiser with his characteristic lucidity managed to pinpoint it:

$$1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 9} + \frac{1}{9 \cdot 61} + \frac{1}{61 \cdot 540} + \frac{1}{540 \cdot 5879} + $$
$$+ \frac{1}{5879 \cdot 75887} + \frac{1}{75887 \cdot 1132426} + \&\text{c.}$$

(1) Here, Lambert commits an error. The following fraction is that of the cot v, while the fraction at the end of the paragraph is that of the tan v.[25] A.S.

Notes to §. 73. Speiser (1946–1948, p. 146).

(1) This fraction is found in *Euler*'s De fractionibus continuis dissertatio,[26] § 30 See *Euler*'s Opera Omnia, series I, vol. 14, pg. 210. A.S.

Notes to §. 88. Speiser (1946–1948, p. 157).

(1) It is possible to untangle these slightly enigmatic statements: The equation between u and v is

$$\frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{1}{i}\frac{e^{iv} - e^{-iv}}{e^{iv} + e^{-iv}} \quad \text{or} \quad \text{tang.hyp}\, u = \text{tang}\, v$$

Putting iv in the place of u and iu in the place of v, the equation remains unaltered. These two formulas

$$\text{tang.hyp}\, u = \text{tang}\, v \qquad y \qquad \text{tang.hyp}\,(iv) = \text{tang}\,(iu)$$

are therefore equivalent.[27] A.S.

[25] The interested reader can easily verify that this is effectively the continued fraction for the tangent by just calculating the convergents:

$$\frac{1}{(w-1)+\frac{1}{1}}, \ \frac{1}{(w-1)+\frac{1}{1+\frac{1}{(3w-2)+\frac{1}{1}}}}, \text{ etc.}$$

Lambert included more convergents because he truncated the continued fraction as follows:

$$\frac{1}{w-1}, \ \frac{1}{(w-1)+\frac{1}{1}}, \ \frac{1}{(w-1)+\frac{1}{1+\frac{1}{(3w-2)}}}, \ \frac{1}{(w-1)+\frac{1}{1+\frac{1}{(3w-2)+\frac{1}{1}}}}, \text{ etc.}$$

On the other hand, due to the fact that:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} = a_1 + \cfrac{1}{a_2 + \ddots}$$

it is easy to get convinced how Lambert passed from one to the other since the tangent and the cotangent are inverse each other.

[26] There is an English translation Euler (1744) (that continued fraction is included in the paragraph 30).

[27] And if these two formulas are equivalent, Lambert can avoid dealing with imaginary ratios (see Barnett (2004, p. 24)).

Appendix D
Echegaray's *Disertaciones Matemáticas Sobre la Cuadratura Del Círculo*

José Echegaray y Eizaguirre (1832–1916) was an Spanish polimath who developed his intelectual life principally in the second part of the 19th century. Although he was an engineer by training, he devoted a great part of his time to literature, becoming Nobel Prize in Literature (1904), acquiring a high reputation in this time in Spain. He also had a rich career in government, being appointed Minister of Education, of Public Works and of Finance successively between 1867 and 1874. Knowing all this, it is noteworthy that mathematics was, in fact, his greatest interest as he remarked often. Santiago Ramón y Cajal, Nobel Prize in Medicine, would say that «he was, without question, the finest and most exquisitely organized brain of 19 Century Spain».[28]

The *Escuela de Ingenieros de Caminos* where Echegaray graduated as the first in his promotion, was at that time, along with the rest of Spanish's *Special Schools in Engineering*, the main school responsable for mathematical education as had happened with France's *Écoles*, upon the model of which Spanish's *Schools* had taken form. This influence is reflected, for instance, in the fact that one of the subjects to be passed by students in order to get into the *Escuela* was «Translate from French», influence that will persist in Echegaray, who will rely to great extent on French authors or French translations.[29]

[28] As for Echegaray I rely heavely on (Sánchez Ron 2004) (quote at p. 602). This paper gave rise to the book of the same author: *José Echegaray (1832–1916) el hombre polifacético. Técnica, ciencia, política y teatro en España* (2016). Fundación Juanelo Turriano.

[29] And actually this influence depended also on France journals, particularly, on Liouville's *Journal*, which is on the other hand reasonable due to its enormous impact. Echegaray probably could not read German (Sánchez Ron 2004, p. 668), so that Crell's *Journal* might have little to do with his mathematical formation.

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As for mathematics, Echegaray wrote around 1913–1915 in his autobiography *Recuerdos*[30] (3 vol., Madrid):

> Mathematics was and is one of my great interests [...] But the cultivation of High Mathematics does not give enough to live [...] I have never abandoned, not even in the most agitated parts of my life, the science of my predilection: but I have never been devoted myself to it as I had wanted.[31]

Without having been an original mathematician, Echegray was a key figure in the introduction of modern European mathematics into Spain. In this sense he was quite connected with the most groundbreaking investigations carried out abroad, studying Gauss' *D.A.* already in the 1850s, giving lectures about Galois' theory, and producing books on several topics ranging from mathematical physics to pure mathematics, his favorite branch.[32]

One of these books was *Disertaciones matemáticas sobre la cuadratura del círculo*[33] with subtitle *Wantzel's method and the division of the circumference into equal parts* (1887), the first chapter of which —*About the impossibility of squaring the circle*— had been published in 1886. This means that Echegaray was presenting a didactical approach to the circle-squaring problem just four years after its solution, which additionaly shows the inmediate influence of this problem we commented elsewhere, being the first introduction of this topic in Spain.

Echegaray did not have access to Lindemann's orignial proof as he himself lamented, having heard of Lindemann's investigation by means of Rouché and Comberousse's *Traité de géométrie élémentaire* (Vol. 1, 5th edition). Echegaray quoted the brief outline to this long-standing problem given by these two authors —in which, by the way, they made reference to both Lambert's proof of the irrationality of π and Legendre's proof or the irrationality of π^2— at the end of which he stated that he aspired to the same objective, namely, to expose Hermite's formulae and theorems, and Lindemann's demonstration. The outcome of this aspiration is a book divided principally into three main parts, gathering together the most recent investigations around this subject:

- *About the impossibility of squaring the circle* Echegaray (1887, pp. 1–49), where Echegaray upon presenting the needed mathematical scafolding (mainly Hermite's formulae and theorems), closed this part with Lindemann's theorem and the impossibility of squaring the circle.

[30] *Remembrances*.

[31] Quoted in (Sánchez Ron 2004, p. 613).

[32] See (Sánchez Ron 2004, pp. 604, 610, 614–615).

[33] *Mathematical dissertations about the circle-squaring problem.*

- *Wantzel's method in order to figure out if a problem can be solved by the straight line and circle method* Echegaray (1887, pp. 50–96), ending with the impossibility of both the problem of doubling the cube and trisecting an angle.
- *Division of the circumference into equal parts* Echegaray (1887, pp. 97–149), following Gauss' achievements in the celebrated last part of D.A., the one that had been immediately admired by mathematicians all over Europe.

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