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Herbert Alexander
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Several Complex Variables and Banach Algebras

Third Edition

Springer

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to Susan and to the memory of Kerstin

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Preface to the Second Edition

During the past twenty years many connections have been found between the theory of analytic functions of one or more complex variables and the study of commutative Banach algebras. On the one hand, function theory has been used to answer algebraic questions such as the question of the existence of idempotents in a Banach algebra. On the other hand, concepts arising from the study of Banach algebras such as the maximal ideal space, the Šilov boundary, Gleason parts, etc. have led to new questions and to new methods of proof in function theory.

Roughly one third of this book is concerned with developing some of the principal applications of function theory in several complex variables to Banach algebras. We presuppose no knowledge of several complex variables on the part of the reader but develop the necessary material from scratch. The remainder of the book deals with problems of uniform approximation on compact subsets of the space of n complex variables. For $n > 1$ no complete theory exists but many important particular problems have been solved.

Throughout, our aim has been to make the exposition elementary and self-contained. We have cheerfully sacrificed generality and completeness all along the way in order to make it easier to understand the main ideas.

Relationships between function theory in the complex plane and Banach algebras are only touched on in this book. This subject matter is thoroughly treated in A. Browder's *Introduction to Function Algebras*, (W. A. Benjamin, New York, 1969) and T. W. Gamelin's *Uniform Algebras*, (Prentice-Hall, Englewood Cliffs, N.J., 1969). A systematic exposition of the subject of uniform algebras including many examples is given by E. L. Stout, *The Theory of Uniform Algebras*, (Bogden and Quigley, Inc., 1971).

The first edition of this book was published in 1971 by Markham Publishing Company. The present edition contains the following new Sections: 18. Submanifolds of High Dimension, 19. Generators, 20. The Fibers Over a Plane Domain, 21. Examples of Hulls. Also, Section 11 has been revised.

Exercises of varying degrees of difficulty are included in the text and the reader should try to solve as many of these as he can. Solutions to starred exercises are given in Section 22.

In Sections 6 through 9 we follow the developments in Chapter 1 of R. Gunning amd H. Rossi, *Analytic Functions of Several Complex Variables*, (Prentice-Hall, Englewood Cliffs, N.J., 1965) or in Chapter III of L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, (Van Nostrand Reinhold, New York, 1966).

I want to thank Richard Basener and John O'Connell, who read the original manuscript and made many helpful mathematical suggestions and improvements. I am also very much indebted to my colleagues, A. Browder, B. Cole, and B. Weinstock for valuable comments. Warm thanks are due to Irving Glicksberg. I am very grateful to Jeffrey Jones for his help with the revised manuscript.

Mrs. Roberta Weller typed the original manuscript and Mrs. Hildegarde Kneisel typed the revised version. I am most grateful to them for their excellent work.

Some of the work on this book was supported by the National Science Foundation.

John Wermer
Providence, R.I.
June, 1975

Preface to the Revised Edition

The second edition of *Banach Algebras and Several Complex Variables*, by John Wermer, appeared in 1976. Since then, there have been many interesting new developments in the subject. The new material in this edition gives an account of some of this work.

We have kept much of the material of the old book, since we believe it to be useful to anyone beginning a study of the subject. In particular, the first ten chapters of the book are unchanged.

Chapter 11 is devoted to maximum modulus algebras, a class of spaces that allows a uniform treatment of several different parts of function theory.

Chapter 12 applies the results of Chapter 11 to uniform approximation by polynomials on curves and arcs in $\mathbb{C}^n$.

Integral kernels in several complex variables generalizing the Cauchy kernel were introduced by Martinelli and Bochner in the 1940s and extended by Leray, Henkin, and others. These kernels allow one to generalize powerful methods in one complex variable based on the Cauchy integral to several complex variables. In Chapter 13, we develop some basic facts about integral kernels, and then in Chapter 14 we give an application to polynomial approximation on compact sets in $\mathbb{C}^n$. Later, in Chapter 19, a different application is given to the problem of constructing a complex manifold with a prescribed boundary.

Chapter 21 studies geometric properties of polynomial hulls, related to area, and Chapter 22 treats topological properties of such hulls. Chapter 23 is concerned with relationships between pseudoconvexity and polynomial hulls, and between pseudoconvexivity and maximum modulus algebras.

A theme that is pursued throughout much of the book is the question of the existence of analytic structure in polynomial hulls. In Chapter 24, several key examples concerning such structures are discussed, both healthy and pathological.

At the end of most of the sections, we have given some historical notes, and we have combined sketches of some of the history of the material of Chapters 11, 12, 20, and 23 in Chapter 25. In addition to keeping the old bibliography of the Second Edition we have included a substantial "Additional Bibliography."

Several other special topics treated in the previous edition are kept in the present version: Chapters 16 and 17 deal with Hörmander's theory of the $\bar{\partial}$-equation in weighted L^2-spaces, and the application of this theory to questions of uniform approximation.

Chapter 18 is concerned with the existence of "Bishop disks," that is, analytic disks whose boundaries lie on a given smooth real submanifold of $\mathbb{C}^n$, and near a point of that submanifold.

Chapter 15 presents the Arens-Royden Theorem on the first cohomology group of the maximal ideal space of a Banach Algebra.

The Appendix gives references for a number of classical results we have used, without proof, in the text.

It is a pleasure to thank Norm Levenberg for his very helpful comments. Thanks also to Marshall Whittlesey.

Herbert Alexander and John Wermer
January 1997

1

Preliminaries and Notation

Let X be a compact Hausdorff space.

- $C_R(X)$ is the space of all real-valued continuous functions on X.
- $C(X)$ is the space of all complex-valued continuous functions on X. By a *measure* μ on X we shall mean a complex-valued Baire measure of finite total variation on X.
- $|\mu|$ is the positive total variation measure corresponding to μ.
- $\|\mu\|$ is $|\mu|(X)$
- $\mathbb{C}$ is the complex numbers.
- $\mathbb{R}$ is the real numbers.
- $\mathbb{Z}$ is the integers.
- $\mathbb{C}^n$ is the space on n-tuples of complex numbers.

Fix n and let Ω be an open subset of $\mathbb{C}^n$.

- $C^k(\Omega)$ is the space of k-times continuously differentiable functions on Ω, $k = 1, 2, \ldots, \infty$.
- $C_0^k(\Omega)$ is the subset of $C^k(\Omega)$ consisting of functions with compact support contained in Ω.
- $H(\Omega)$ is the space of holomorphic functions defined on Ω.

By *Banach algebra* we shall mean a commutative Banach algebra with unit. Let $\mathfrak{A}$ be such an object.

- $\mathcal{M}(\mathfrak{A})$ is the space of maximal ideals of $\mathfrak{A}$. When no ambiguity arises, we shall write $\mathcal{M}$ for $\mathcal{M}(\mathfrak{A})$. If m is a homomorphism of $\mathfrak{A} \to \mathbb{C}$, we shall frequently identifiy m with its kernel and regard m as an element of $\mathcal{M}$. For f in $\mathfrak{A}$, M in $\mathcal{M}$,
- $\hat{f}(M)$ is the value at f of the homomorphism of $\mathfrak{A}$ into $\mathbb{C}$ corresponding to M. We shall sometimes write $f(M)$ instead of $\hat{f}(M)$.
- $\hat{\mathfrak{A}}$ is the algebra consisting of all functions $\hat{f}$ on $\mathcal{M}$ with f in $\mathfrak{A}$. For x in $\mathfrak{A}$,
- $\sigma(x)$ is the spectrum of $x = \{\lambda \in \mathbb{C} \mid \lambda - x$ has no inverse in $\mathfrak{A}\}$.
- $\operatorname{rad}\mathfrak{A}$ is the radical of $\mathfrak{A}$. For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$,
- $|z| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$.

For S a subset of a topological space,

- $\dot{S}$ is the interior of S,

$\bar{S}$ is closure of S, and
∂S is the boundary of S.
For X a compact subset of $\mathbb{C}^n$,
$P(X)$ is the closure in $C(X)$ of the polynomials in the coordinates.
Let Ω be a plane region with compact closure $\bar{\Omega}$. Then
$A(\Omega)$ is the algebra of all functions continuous on $\bar{\Omega}$ and holomorphic on Ω.

Let X be a compact space, $\mathcal{L}$ a subset of $C(X)$, and μ a measure on X. We write $\mu \perp \mathcal{L}$ and say μ is *orthogonal* to $\mathcal{L}$ if

$$\int f \, d\mu = 0 \quad \text{for all } f \text{ in } \mathcal{L}$$

We shall frequently use the following result (or its real analogue) without explicitly appealing to it:

Theorem (Riesz-Banach). *Let $\mathcal{L}$ be a linear subspace f $C(X)$ and fix g in $C(X)$. If for every measure μ on X*

$$\mu \perp \mathcal{L} \textit{ implies } \mu \perp g,$$

then g lies in the closure of $\mathcal{L}$. In particular, if

$$\mu \perp \mathcal{L} \textit{ implies } \mu = 0,$$

then $\mathcal{L}$ is dense in C(X).

We shall need the following elementary fact, left to the reader as

EXERCISE 1.1. Let X be a compact space. Then to every maximal ideal M of $C(X)$ corresponds a point x_0 in X such that $M = \{f \text{ in } C(X) | f(x_0) = 0\}$. Thus $\mathcal{M}(C(X)) = X$.

Here are some example of Banach algebras.

(a) Let T be a bounded linear operator on a Hilbert space H and let $\mathfrak{A}$ be the closure in operator norm on H of all polynomials in T. Impose the operator norm on $\mathfrak{A}$.

(b) Let $C^1(a, b)$ denote the algebra of all continuously differentiable functions on the interval [a,b], with

$$\|f\| = \max_{[a,b]} |f| + \max_{[a,b]} |f'|.$$

(c) Let Ω be a plane region with compact closure $\bar{\Omega}$. Let $A(\Omega)$ denote the algebra of all functions continuous on $\bar{\Omega}$ and holomorphic in Ω, with

$$\|f\| = \max_{z \in \bar{\Omega}} |f(z)|.$$

(d) Let X be a compact subset of C^n. Denote by $P(X)$ the algebra of all functions defined on X which can be approximated by polynomials in the coordinates $z_1, \ldots, z_n$ uniformly on X, with

$$\|f\| = \max_{x} |f|.$$

(e) Denote by $H^\infty(D)$ the algebra of all bounded holomorphic functions defined in the open unit disk D. Put

$$\|f\| = \sup_D |f|.$$

(f) Let X be a compact subset of the plane. $R(X)$ denotes the algebra of all functions on X which can be uniformly approximated on X by functions holomorphic in some neighborhood of X. Take

$$\|f\| = \max_x |f|.$$

(g) Let X be a compact Hausdorff space. On the algebra $C(X)$ of all complex-valued continuous functions on X we impose the norm

$$\|f\| = \max_x |f|.$$

Definition. Let X be a compact Hausdorff space. A *uniform algebra* on X is an algebra $\mathfrak{A}$ of continuous complex-valued functions on X satisfying

(i) $\mathfrak{A}$ is closed under uniform convergence on X.
(ii) $\mathfrak{A}$ contains the constants.
(iii) $\mathfrak{A}$ separates the points of X

$\mathfrak{A}$ is normed by $\|f\| = \max_x |f|$ and so becomes a Banach algebra.

Note that $C(X)$ is a uniform algebra on X, and that every other uniform algebra on X is a proper closed subalgebra of $C(X)$. Among our examples, (c), (d), (f), and (g) are uniform algebras; (a) is not, except for certain T, and (b) is not.

If $\mathfrak{A}$ is a uniform algebra, then clearly

$$\|x^2\| = \|x\|^2 \quad \text{for all } x \in \mathfrak{A}. \tag{1}$$

Conversely, let $\mathfrak{A}$ be a Banach algebra satisfying (1). We claim that $\mathfrak{A}$ is isometrically isomorphic to a uniform algebra. For (1) implies that

$$\|x^4\| = \|x\|^4, \dots, \|x^{2^n}\| = \|x\|^{2^n}, \quad \text{all } n.$$

Hence

$$\|x\| = \lim_{k\to\infty} \|x^k\|^{1/k} = \max_{\mathcal{M}} |\hat{x}|.$$

Since $\mathfrak{A}$ is complete in its norm, it folows that $\hat{\mathfrak{A}}$ is complete in the uniform norm on $\mathcal{M}$, so $\hat{\mathfrak{A}}$ is closed under uniform convergence on $\mathcal{M}$. Hence $\hat{\mathfrak{A}}$ is a uniform algebra on $\mathcal{M}$ and the map $x \to \hat{x}$ is an isometric isomorphism from $\mathfrak{A}$ to $\hat{\mathfrak{A}}$.

It follows that the algebra $H^\infty(D)$ of example (e) is isometrically isomorphic to a uniform algebra on a suitable compact space.

In the later portions of this book, starting with Section 10, we shall study uniform algebras, whereas the earlier sections (as well as Section 15) will be concerned with arbitrary Banach algebras.

Throughout, when studying general theorems, the reader should keep in mind some concrete examples such as those listed under (a) through (g), and he should make clear to himself what the general theory means for the particular examples.

EXERCISE 1.2. Let $\mathfrak{A}$ be a uniform algebra on X and let h be a homomorphism of $\mathfrak{A} \to \mathbb{C}$. Show that there exists a probability measure (positive measure of total mass 1) μ on X so that

$$h(f) = \int_X f\, d\mu, \quad \text{all } f \text{ in } \mathfrak{A}.$$

2

Classical Approximation Theorems

Let X be a compact Hausdorff space. Let $\mathfrak{A}$ be a subalgebra of $C_R(X)$ which contains the constants.

Theorem 2.1 (Real Stone-Weierstrass Theorem). *If $\mathfrak{A}$ separates the points of X, then $\mathfrak{A}$ is dense in $C_R(X)$.*

We shall deduce this result from the following general theorem:

Proposition 2.2. *Let B be a real Banach space and B^* its dual space taken in the weak-* topology. Let K be a nonempty compact convex subset of B^*. Then K has an extreme point.*

Note. If W is a real vector space, S a subset of W, and p a point of S, then p is called an *extreme point* of S provided

$$p = \tfrac{1}{2}(p_1 + p_2), \qquad p_1, p_2 \in S \Rightarrow p_1 = p_2 = p.$$

If S is a convex set and p an extreme point of S, then $0 < \theta < 1$ and $p = \theta p_1 + (1 - \theta)p_2$ implies that $p_1 = p_2 = p$.

We shall give the proof for the case that B is separable.

PROOF. Let $\{L_n\}$ be a countable dense subset of B. If $y \in B^*$, put

$$L_n(y) = y(L_n).$$

Define

$$l_1 = \sup_{x \in K} L_1(x).$$

Since K is compact and L_1 continuous, l_1 is finite and attained; i.e., $\exists x_1 \in K$ with $L_1(x_1) = l_1$. Put

$$l_2 = \sup L_2(x) \text{ over all } x \in K, \qquad \text{with} \qquad L_1(x) = l_1.$$

Again, the sup is taken over a compact set, contained in K, so $\exists x_2 \in K$ with

$$L_2(x_2) = l_2 \qquad \text{and} \qquad L_1(x_2) = l_1.$$

Going on in this way, we get a sequence $x_1, x_2, \ldots$ in K so that for each n.

$$L_1(x_n) = l_1,\ L_2(x_n) = l_2, \ldots, L_n(x_n) = l_n,$$

and

$$l_{n+1} = \sup L_{n+1}(x) \text{ over } x \in K \qquad \text{with} \qquad L_1(x) = l_1, \ldots, L_n(x) = l_n.$$

Let x^* be an accumulation point of $\{x_n\}$. Then $x^* \in K$.
$L_j(x_n) = l_j$ for all large n. So $L_j(x^*) = l_j$ for all j.
We claim that x^* is an extreme point in K. For let

$$x^* = \tfrac{1}{2} y_1 + \tfrac{1}{2} y_2, \qquad y_1, y_2 \in K.$$
$$l_1 = L_1(x^*) = \tfrac{1}{2} L_1(y_1) + \tfrac{1}{2} L_1(y_2).$$

Since

$$L_1(y_j) \leq l_1,\ j = 1, 2,\ L_1(y_1) = L_1(y_2) = l_1.$$

Also,

$$l_2 = L_2(x^*) = \tfrac{1}{2} L_2(y_1) + \tfrac{1}{2} L_2(y_2).$$

Since $L_1(y_1) = l_1$ and $y_1 \in K$, $L_2(y_1) \leq l_2$. Similarly, $L_2(y_2) \leq l_2$. Hence

$$L_2(y_1) = L_2(y_2) = l_2.$$

Proceeding in this way, we get

$$L_k(y_1) = L_k(y_2) \qquad \text{for all } k.$$

But $\{L_k\}$ was dense in B. It follows that $y_1 = y_2$. Thus x^* is extreme in K. □

Note. Proposition 2.2 (without separability assumption) is proved in [23, pp. 439-440]. In the application of Proposition 2.2 to the proof of Theorem 2.1 (see below), $C_R(X)$ is separable provided X is a metric space.

PROOF OF THEOREM 2.1. Let

$$K = \{\mu \in (C_R(X))^* | \mu \perp \mathfrak{A} \text{ and } \|\mu\| \leq 1\}.$$

K is a compact, convex set in $(C_R(X))^*$. (Why?) Hence K has an extreme point σ, by Proposition 2.2. Unless $K = \{0\}$, we can choose σ with $\|\sigma\| = 1$. Since $1 \in \mathfrak{A}$ and so

$$\int 1 \, d\sigma = 0,$$

σ cannot be a point mass and so $\exists$ distinct points x_1 and x_2 in the carrier of σ.

Choose $g \in \mathfrak{A}$ with $g(x_1) \neq g(x_2)$, $0 < q < 1$. (How?) Then

$$\sigma = g \cdot \sigma + (1-g)\sigma = \|g\sigma\| \frac{g\sigma}{\|g\sigma\|} + \|(1-g)\sigma\| \frac{(1-g)\sigma}{\|(1-g)\sigma\|}.$$

Also,

$$\|g\sigma\| + \|(1-g)\sigma\| = \int g\, d|\sigma| + \int (1-g)\, d|\sigma| = \int d|\sigma| = \|\sigma\| = 1.$$

Thus σ is a convex combination of $g\sigma/\|g\sigma\|$ and $(1-g)\sigma/\|(1-g)\sigma\|$. But both of these measures lie in K. (Why?) Hence

$$\sigma = \frac{g\sigma}{\|g\sigma\|}.$$

It follows that g is constant a.e. - $d|\sigma|$. But $g(x_1) \neq g(x_2)$ and g is continuous which gives a contradiction.

Hence $K = \{0\}$ and so $\mu \in (C_R(X))^*$ and $\mu \perp \mathfrak{A} \Rightarrow \mu = 0$. Thus $\mathfrak{A}$ is dense in $C_R(X)$, as claimed.

Theorem 2.3 (Complex Stone-Weierstrass Theorem). *$\mathfrak{A}$ is a subalgebra of $C(X)$ containing the constants and separating points. If*

(1) $$f \in \mathfrak{A} \Rightarrow \bar{f} \in \mathfrak{A},$$

then $\mathfrak{A}$ is dense in $C(X)$.

Proof. Let $\mathcal{L}$ consists of all real-valued functions in $\mathfrak{A}$. Since by (1) $\mathcal{L}$ contains Re f and Im f for each $f \in \mathfrak{A}$, $\mathcal{L}$ separates points on X. Evidently $\mathcal{L}$ is a subalgebra of $C_R(X)$ containing the (real) constants. By Theorem 2.1 $\mathcal{L}$ is then dense in $C_R(X)$. It follows that $\mathfrak{A}$ is dense in $C(X)$. (How?)

Let Σ_R denote the real subspace of $C^n = \{(z_1, \ldots, z_n) \in C^n | z_j \text{ is real, all } j\}$.

Corollary 1. *Let X be a compact subset of Σ_R. Then $P(X) = C(X)$.*

Proof. Let $\mathfrak{A}$ be the algebra of all polynomials in $z_1, \ldots, z_n$ restricted to X. $\mathfrak{A}$ then satisfies the hypothesis of the last theorem, and so $\mathfrak{A}$ is dense in $C(X)$; i.e., $P(X) = C(X)$.

Corollary 2. *Let I be an interval on the real line. Then $P(I) = C(I)$.*

This is, of course, the Weierstrass approximation theorem (slightly complexified).

Let us replace I by an arbitrary compact subset X of $\mathbb{C}$. When does $P(X) = C(X)$? It is easy to find necessary conditions on X. (Find some.) However, to get a complete solution, some machinery must first be built up.

The machinery we shall use will be some elementary potential theory for the Laplace operator Δ in the plane, as well as for the Cauchy-Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right).$$

These general results will then be applied to several approximation problems in the plane, including the above problem of characterizing those X for which $P(X) = C(X)$.

Let μ be a measure of compact support $\subset \mathbb{C}$. We define the *logarithmic potential* μ^* of μ by

$$(2) \qquad \mu^*(z) = \int \log \left| \frac{1}{z-\zeta} \right| d\mu(\zeta).$$

We define the *Cauchy transform* $\hat{\mu}$ of μ by

$$(3) \qquad \hat{\mu}(z) = \int \frac{1}{\zeta - z} d\mu(\zeta).$$

Lemma 2.4. *The functions*

$$\int \left| \log \left| \frac{1}{z-\zeta} \right| \right| d|\mu|(\zeta) \qquad \text{and} \qquad \int \left| \frac{1}{\zeta - z} \right| d|\mu|(\zeta)$$

are summable - $dx\,dy$ over compact sets in $\mathbb{C}$. It follows that these functions are finite a.e. *- $dx\,dy$ and hence that μ^* and $\hat{\mu}$ are defined* a.e. *- $dx\,dy$.*

Since $1/r \geq |\log r|$ for small $r > 0$, we need only consider the second integral. Fix $R > 0$ with $\operatorname{supp}|\mu| \subset \{z \big| |z| < R\}$.

$$\gamma = \int_{|z| \leq R} dx\,dy \left\{ \int \left| \frac{1}{\zeta - z} \right| d|\mu|(\zeta) \right\} = \int d|\mu|(\zeta) \int_{|z| \leq R} \frac{dx\,dy}{|z-\zeta|}.$$

For $\zeta \in \operatorname{supp}|\mu|$ and $|z| \leq R$, $|z - \zeta| \leq 2R$.

$$\int_{|z| \leq R} \frac{dx\,dy}{|z-\zeta|} \leq \int_{|z'| \leq 2R} \frac{dx'\,dy'}{|z'|} = \int_0^{2R} r\,dr \int_0^{2\pi} \frac{d\theta}{r} = 4\pi R.$$

Hence $\gamma \leq 4\pi R \cdot \|\mu\|$. $\square$

Lemma 2.5. *Let $F \in C_0^1(\mathbb{C})$. Then*

$$(4) \qquad F(\zeta) = -\frac{1}{\pi} \int_{\mathbb{C}}\int \frac{\partial F}{\partial \bar{z}} \frac{dx\,dy}{z - \zeta}, \qquad \text{all } \zeta \in C.$$

Note. The proof uses differential forms. If this bothers you, read the proof after reading Sections 4 and 5, where such forms are discussed, or make up your own proof.

Proof. Fix ζ and choose $R > |\zeta|$ with $\operatorname{supp} F \subset \{z \big| |z| < R\}$. Fix $\varepsilon > 0$ and small. Put $\Omega_\varepsilon = \{\big| |z| < R \text{ and } |z - \zeta| > \varepsilon\}$.

The 1-form $F\,dz/z - \zeta$ is smooth on Ω_ε and

$$d\left(\frac{F\,dz}{z-\zeta}\right) = \frac{\partial}{\partial \bar{z}} \left(\frac{F}{z-\zeta}\right) d\bar{z} \wedge dz = \frac{\partial F}{\partial \bar{Z}} \frac{d\bar{z} \wedge dz}{z - \zeta}.$$

By Stokes's theorem

$$\int_{\Omega_\varepsilon} d\left(\frac{F\,dz}{z-\zeta}\right) = \int_{\partial\Omega_\varepsilon} \frac{F\,dz}{z-\zeta}.$$

Since $F = 0$ on $\{z \big| |z| = R\}$, the right side is

$$\int_{|z-\zeta|=\varepsilon} \frac{F\,dz}{z-\zeta} = -\int_0^{2\pi} F(\zeta+\varepsilon e^{i\theta})i\,d\theta,$$

so

$$\int_{\Omega_\varepsilon} \frac{\partial F}{\partial \bar z}\frac{d\bar z\wedge dz}{z-\zeta} = -\int_o^{2\pi} F(\zeta+\varepsilon e^{i\theta})i\,d\theta.$$

Letting $\varepsilon \to 0$ we get

$$\int_{|z|<R} \frac{\partial F}{\partial \bar z}\frac{d\bar z\wedge dz}{z-\zeta} = -2\pi i F(\zeta).$$

Since $\partial F/\partial \bar z$ for $|z| > R$ and since $d\bar z \wedge dz = 2i\,dx \wedge dy$, this gives

$$\int \frac{\partial F}{\partial \bar z}\frac{dx\,dy}{z-\zeta} = -\pi F(\zeta),$$

i.e., (4).

Note. The intuitive content of (4) is that arbitrary smooth functions can be synthesized from functions

$$f_\delta(\zeta) = \frac{1}{\lambda-\zeta}$$

by taking linear combinations and then limits.

Lemma 2.6. *Let $G \in C_0^2(\mathbb{C})$. Then*

$$G(\zeta) = -\frac{1}{2\pi}\int_C\int \Delta G(z)\log\frac{1}{|z-\zeta|}\,dx\,dy, \qquad \textit{all } \zeta \in \mathbb{C}. \tag{5}$$

PROOF. The proof is very much like that of Lemma 2.5. With Ω_ε as in that proof, start with Green's formula

$$\int_{\Omega_\varepsilon}\int (u\Delta v - v\Delta u)\,dx\,dy = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds$$

and take $u = G$, $v = \log|z-\zeta|$. We leave the details to you.

Lemma 2.7. *If μ is a measure with compact support in C, and if $\hat\mu(z) = 0$ a.e. $-$ $dx\,dy$, then $\mu = 0$. Also, if $\mu^*(z) = 0$ a.e. $-$ $dx\,dy$, then $\mu = 0$.*

PROOF. Fix $g \in C_0^1(\mathbb{C})$. By (4)

$$\int g(\zeta)\,d\mu(\zeta) = \int d\mu(\zeta)\left[-\frac{1}{\pi}\int \frac{\partial g}{\partial \bar Z}(z)\frac{dx\,dy}{z-\zeta}\right].$$

Fubini's theorem now gives

$$\frac{1}{\pi}\int \frac{\partial g}{\partial \bar{z}}(z)\hat{\mu}(z)\,dx\,dy = \int g\,d\mu. \tag{6}$$

Since $\hat{\mu} = 0$ a.e., we deduce that

$$\int g\,d\mu = 0.$$

But the class of functions obtained by restricting to supp μ the functions in $C_0^1(\mathbb{C})$ is dense in $C(\text{supp }\mu)$ by the Stone-Weierstrass theorem. Hence $\mu = 0$.

Using (5), we get similarly for $g \in C_0^2(\mathbb{C})$,

$$-\int g\,d\mu = \frac{1}{2\pi}\int \Delta g(z)\cdot \mu^*(z)\,dx\,dy$$

and conclude that $\mu = 0$ if $\mu^* = 0$ a.e.

As a first application, consider a compact set $X \subset C$.

Theorem 2.8 (Hartogs-Rosenthal). *Assume that X has Lebesgue two-dimensional measure 0. Then rational functions whose poles lie off X are uniformly dense in $C(X)$.*

PROOF. Let W be the linear space consisting of all rational functions holomorphic on X. W is a subspace of $C(X)$. To show W dense, we consider a measure μ on X with $\mu \perp W$. Then $\hat{\mu}(z) = \int d\mu(\zeta)/\zeta - z = 0$ for $z \notin X$, since $1/\zeta - z \in W$ for such z. and $\mu \perp W$.

Since X has measure 0, $\hat{\mu} = 0$ a.e. $-dx\,dy$. Lemma 2.7 yields $\mu = 0$.

Hence $\mu \perp W \Rightarrow \mu = 0$ and so W is dense. □

As a second application, consider an open set $\Omega \subset \mathbb{C}$ and a compact set $K \subset \Omega$. (In the proofs of the next two theorems we shall supposed Ω biunded and leave the modifications for the genereal case to the reader.)

Theorem 2.9 (Runge). *If F is a holomorphic function defined on Ω, there exists a sequence $\{R_n\}$ of rational functions holomorphic in Ω with*

$$R_n \to F \textit{ uniformly on } K.$$

PROOF. Let $\Omega_1, \Omega_2, \ldots$ be the components of $\mathbb{C}\backslash K$. It is no loss of generality to assume that each Ω_j meets the complement of Ω. (Why?) Fix $p_i \in \Omega_j\backslash\Omega$.

Let W be the space of all rational functions regular except for the possible poles at some of the p_j, restricted to K. Then W is a subspace of $C(K)$ and it suffices to show that W contains F in its closure.

Choose a measure μ on K with $\mu \perp W$. We must show that $\mu \perp F$.

Fix $\phi \in C^\infty(\mathbb{C})$, supp $\phi \subset \Omega$ and $\phi = 1$ in a neighborhood N of K.

Using (6) with $g = F\cdot\phi$ we get

$$\frac{1}{n}\int \frac{\partial(F\phi)}{\partial z}(z)\hat{\mu}(z)\,dx\,dy = \int F\phi d\mu. \tag{7}$$

Fix j.

$$\hat{\mu}(z) = \int \frac{d\mu(\zeta)}{\zeta - z}$$

is analytic in Ω_j and

$$\frac{d^k \hat{\mu}}{dz^k}(p_j) = k! \int \frac{d\mu(\zeta)}{(\zeta - p_j)^{k+1}}, \qquad k = 0, 1, 2, \ldots.$$

The right-hand side is 0 since $(\zeta - p_j)^{-(k+1)} \in W$ and $\mu \perp W$. Thus all derivatives of $\hat{\mu}$ vanish at p_j and hence $\hat{\mu} = 0$ in Ω_j. Thus $\hat{\mu} = 0$ on $\mathbb{C}\backslash K$. Also, $F\phi = F$ is analytic in N, and so

$$\frac{\partial}{\partial \bar{z}}(F\phi) = 0 \text{ on } K.$$

The integrand on the left in (7) thus vanishes everywhere, and so

$$\int F\, d\mu = \int F\Phi\, d\mu = 0.$$

Thus $\mu \perp W \Rightarrow \mu \perp F$. □

When can we replace "rational function" by "polynomial" in the last theorem? Suppose that Ω is *multiply connected*. Then we cannot.

The reason is this: We can choose a simple closed curve β lying in Ω such that some point z_0 in the interior of β lies outside Ω. Put

$$F(z) = \frac{1}{z - z_0}.$$

Then F is holomorphic is Ω. Suppose that $\exists$ a sequence of polynomials $\{P_n\}$ converging uniformly to F on β. Then

$$(z - z_0)P_n - 1 \to \theta \text{ uniformly on } \beta.$$

By the maximum principle

$$(z - z_0)P_n - 1 \to 0 \text{ inside } \beta.$$

But this is false for $z = z_0$.

Theorem 2.10 (Runge). *Let Ω be a simply connected region and fix G holomorphic in Ω. if K is a compact subset of Ω, then $\exists$ a sequence $\{P_n\}$ of polynomials converging uniformly to G on K.*

Proof. Without loss of generality we may assume that $\mathbb{C}\backslash K$ is connected.

Fix a point p in $\mathbb{C}$ lying outside a disk $\{z \big| |z| \le R\}$ which contains K. The proof of the last theorem shows that $\exists$ rational functions R_n with sole pole at p with

$$R_n \to G \text{ uniformly on } K.$$

The Taylor expansion around 0 for R_n converges uniformly on K. Hence we can replace R_n by a suitable partial sum P_n of this Taylor series, getting

$$P_n \to G \text{ uniformly on } K.$$

$\square$

We return now to the problem of describing those compact sets X in the z-plane which satisfy $P(X) = C(X)$.

Let p be an interior point of X. Then every f in $P(X)$ is analytic at p. Hence the condition

$$\text{The interior of } X \text{ is empty.} \tag{8}$$

is necessary for $P(X) = C(X)$.

Let Ω_1 be a bounded component of $\mathbb{C}$ X. Fix $F \in P(X)$. Choose polynomials P_n with

$$P_n \to F \text{ uniformly on } X.$$

Since $\partial\Omega_1 \subset X$,

$$|P_n - P_m| \to 0 \text{ uniformly on } \partial\Omega_1$$

as $n, m \to 0$. Hence by the maximun principle

$$|P_n - P_m| \to 0 \text{ uniformly on } \Omega_1.$$

Hence P_n converges uniformly on $\Omega_1 \cup \partial\Omega_1$ to a function holomorphic on Ω_1, continuous on $\Omega_1 \cup \partial\Omega_1$, and $= F$ on $\partial\Omega_1$.

This restricts the elements F of $P(X)$ to a proper subset of $C(X)$. (Why?) Hence the condition

$$\mathbb{C}\backslash X \text{ is connected.} \tag{9}$$

is also necessary for $P(X) = C(X)$.

Theorem 2.11 (Lavrentieff). *If (8) and (9) hold, then* $P(X) = C(X)$.

Note that the Stone-Weierstrass theorem gives us no help here, for to apply it we should need to know that $\bar{z} \in P(X)$, and to prove that is as hard as the whole theorem.

The chief step in our proof is the demonstration of a certain continuity property of the logarithmic potential α^* of a measure α supported on a compact plane set E with connected complement, as we approach a boundary point z_0 of E from $C\backslash E$.

Lemma 2.12 (Carleson). *Let E be a compact plane set with $\mathbb{C}\backslash E$ connected and fix $z_0 \in \partial E$. Then $\exists$ probability measures σ_t for each $t > 0$ with σ_t carried on $\mathbb{C}\backslash E$ such that:*

Let α be a real measure on E satisfying

$$\int_E \left|\log \left|\frac{1}{z_0 - \zeta}\right|\right| d|\alpha|(\zeta) < \infty. \tag{10}$$

Then

$$\lim_{t \to 0} \int \alpha^* \, d\sigma_t(z) = \alpha^*(z_0).$$

PROOF. We may assume that $z_0 = 0$. Fix $t > 0$. Since $0 \in \partial E$ and $\mathbb{C} \backslash E$ is connected, $\exists$ a probability measure σ_t carried on $\mathbb{C} \backslash E$ such that

$$\sigma_t\{z | r_1 < |z| < r_2\} = \frac{1}{t}(r_2 - r_1) \qquad \text{for } 0 < r_1 < r_2 \leq t$$

and $\sigma_t = 0$ outside $|z| \leq t$.

If some line segment with 0 as one end point and length t happens to lie in $\mathbb{C} \backslash E$, we may of course take σ_t as $1/t$·linear measure on that segment. (In the general case, construct σ_t.)

Then for all $\zeta \in \mathbb{C}$ we have

$$\begin{aligned}
\int \log \left|\frac{1}{z - \zeta}\right| d\sigma_t(z) &\leq \int \log \left|\frac{1}{|z| - |\zeta|}\right| d\sigma_t(z) \\
&= \frac{1}{t} \int_0^t \log \frac{1}{|r| - |\zeta||} \, dr \\
&= \log \frac{1}{|\zeta|} + \frac{1}{t} \int_0^t \log \frac{1}{|1 - r/|\zeta||} \, dr.
\end{aligned}$$

The last term is bounded above by a constant A independent of t and $|\zeta|$. (Why?) Hence we have

$$\int \log \left|\frac{1}{z - \zeta}\right| d\sigma_t(z) \leq \log \frac{1}{|\zeta|} + A, \qquad \text{all } \zeta, \text{ all } t > 0. \tag{11}$$

Also, as $t \to 0$, $\sigma_t \to$ point mass at 0. Hence for each fixed $\zeta \neq 0$.

$$\int \log \left|\frac{1}{z - \zeta}\right| d\sigma_t(z) \to \log \frac{1}{|\zeta|}. \tag{12}$$

Now for fixed t Fubini's theorem gives

$$\int \alpha^*(z) d\sigma_t(z) = \int \left\{\int \log \left|\frac{1}{z - \zeta}\right| d\sigma_t(z)\right\} d\alpha(\zeta). \tag{13}$$

By (11), (12), and (10), the integrand on the right tends to $\log 1/|\zeta|$ dominatedly with respect to $|\alpha|$. Hence the right side approaches

$$\int \log \frac{1}{|\zeta|} \, d\alpha(\zeta) = \alpha^*(0)$$

as $t \to 0$, and so

$$\lim_{t\to 0} \int \alpha^*(z)d\sigma_t(z) = a^*(0).$$

$\square$

Proof of Theorem 2.11. Let α be a real measure on X with $\alpha \perp \mathrm{Re}(P(X))$. Then

$$\int \mathrm{Re}\, \zeta^n d\alpha(\zeta) = 0, \qquad n \geq 0$$

and

$$\int \mathrm{Im}\, \zeta^n d\alpha = \int \mathrm{Re}(-i\zeta^n) d\alpha = 0, \qquad n \geq 0,$$

so that

$$\int \zeta^n d\alpha = 0, \qquad n \geq 0.$$

For $|z|$ large,

$$\log\left(1 - \frac{\zeta}{z}\right) = \sum_0^\infty c_n(z)\zeta^n,$$

the series converging uniformly for $\zeta \in X$. Hence

$$\int \log\left(1 - \frac{\zeta}{z}\right) d\alpha(\zeta) = \sum_0^\infty c_n(z) \int \zeta^n d\alpha(\zeta) = 0,$$

whence

$$\int \mathrm{Re}\left(\log\left(1 - \frac{\zeta}{z}\right)\right) d\alpha(\zeta) = 0$$

or

$$\int \log|z - \zeta| d\alpha(\zeta) - \int \log|z| d\alpha(\zeta) = 0,$$

whence

$$\int \log|z - \zeta| d\alpha(\zeta) = 0,$$

since $\alpha \perp 1$. Since

$$\int \log|z - \zeta| d\alpha(\zeta) = 0$$

is harmonic in $\mathbb{C}\backslash X$, the function vanishes not only for large $|z|$, but in fact for all z in $\mathbb{C}\backslash X$, and so

$$\alpha^*(z) = 0, \qquad z \in \mathbb{C}\backslash X.$$

By Lemma 2.12 it follows that we also have

$$\alpha^*(z_0) = 0, \qquad z_0 \in X,$$

provided (10) holds at z_0. By Lemma 2.4 this implies that

$$\alpha^* = 0 \text{ a.e.} - dx\,dy.$$

By Lemma 2.7 this implies that $\alpha = 0$. Hence

(14) $$\operatorname{Re} P(X) \text{ is dense in } C_R(X).$$

Now choose $\mu \in P(X)^\perp$. Fix $z_0 \in X$ with

(15) $$\int \left| \frac{1}{z - z_0} \right| d|\mu|(z) < \infty.$$

Because of (14) we can find for each positive integer k a polynomial P_k such that

(16) $$|\operatorname{Re} P_k(z) - |z - z_0|| \leq \frac{1}{k}, \qquad z \in X$$

and

(17) $$P_k(z_0) = 0.$$

$$f_k(z) = \frac{e^{-kP_k(z)} - 1}{z - z_0}$$

is an entire function and hence its restriction to X lies in $P(X)$. Hence

(18) $$\int f_k\, d\mu = 0.$$

Equation (16) gives

$$\operatorname{Re} kP_k(z) - k|z - z_0| \geq -1,$$

whence

$$|e^{-kP_k(z)}| \leq e^{-k|z - z_0| + 1}, \qquad z \in X.$$

It follows that $f_k(z) \to -1/z - z_0$ for all $z \in X \backslash \{z_0\}$, as $k \to \infty$, and also

$$|f_k(z)| \leq \frac{4}{|z - z_0|}, \qquad z \in X.$$

Since by (15) $1/|z - z_0|$ is summable with respect to $|\mu|$, this implies that

$$\int f_k\, d\mu \to -\int \frac{d\mu(z)}{z - z_0}$$

by dominated convergence.

Equation (18) then gives that

$$\int \frac{d\mu(z)}{z - z_0} = 0.$$

Since (15) holds a.e. on X by Lemma 2.4, and since certainly

$$\int \frac{d\mu(z)}{z - z_0} = 0 \qquad \text{for } z_0 \in \mathbb{C}\backslash X$$

(why?), we conclude that $\hat{\mu} = 0$ a.e., so $\mu = 0$ by Lemma 2.7. Thus $\mu \perp P(X) \Rightarrow \mu = 0$, and so $P(X) = C(X)$. $\square$

NOTES

Proposition 2.2 is a part of the Krein–Milman theorem [4, p. 440]. The proof of Theorem 2.1 given here is due to de Branges [Bra]. Lemma 2.7 (concerning $\hat{\mu}$) is given by Bishop in [Bi1]. Theorem 2.8 is in F. Hartogs and A. Rosenthal, Über Folgen analytischer Funktionen, *Math. Ann.* **104** (1931). Theorem 2.9 is due to C. Runge, Zur Theorie der eindeutigen analytischen Funktionen, *Acta Math.* **6** (1885). The proof given here is found in [Hö2, Chap. 1]. Theorem 2.11 was proved by M. A. Lavrentieff, *Sur les fonctions d'une variable complexe représentables par des séries de ploynomes*, Hermann, Paris, 1936, and a simpler proof is due to S. N. Mergelyan, On a theorem of M. A. Lavrentieff, *A.M.S. Transl* **86** (1953). Lemma 2.12 and its use in the proof of Theorem 2.11 is in L. Carleson, Mergelyan's theorem on uniform polynomial approximation, *Math. Scand.* **15** (1964), 167–175.

Theorem 2.1 is due to M. H. Stone, Applications of the theory of Boolean rings to general topology, *Trans. Am. Math. Soc.* **41** (1937). See also M. H. Stone, The generalized Weierstrass approximation theorem, *Math. Mag.* **21** (1947–1948).

3

Operational Calculus in One Variable

Let $\mathcal{F}$ denote the algebra of all functions f on $-\pi \leq \theta \leq \pi$, with

$$f(\theta) = \sum_{-\infty}^{\infty} C_n e^{in\theta}, \qquad \sum_{-\infty}^{\infty} |C_n| < \infty,$$

EXERCISE 3.1. $\mathcal{M}(\mathcal{F})$ may be identified with the circle $|\zeta| = 1$ and for $f = \sum_{\infty}^{\infty} C_n e^{in\theta}$, $|\zeta_0| = 1$,

$$\hat{f}(\zeta_0) = \sum_{-\infty}^{\infty} C_n \zeta_0^n.$$

If $f \in \mathcal{F}$ and f never vanishes on $-\pi \leq \theta \leq \pi$, it follows that $\hat{f} \neq 0$ on $\mathcal{M}(\mathcal{F})$ and so that f has an inverse in $\mathcal{F}$, i.e.,

$$\frac{1}{f} = \sum_{-\infty}^{\infty} d_n e^{in\theta}$$

with $\sum_{-\infty}^{\infty} |d_n| < \infty$.

This result, that nonvanishing elements of $\mathcal{F}$ have inverses in $\mathcal{F}$, is due to Wiener (see [Wi, p. 91]), by a quite different method.

We now ask: Fix $f \in \mathcal{F}$ and let σ be the range of f; i.e.,

$$\sigma = \left\{ f(\theta) | - \pi \leq \theta \leq \pi \right\}.$$

Let Φ be a continuous function defined on σ, so that $\Phi(f)$ is a continuous function on $[-\pi, \pi]$. Does $\Phi(f) \in \mathcal{F}$?

The preceding result concerned the case $\Phi(z) = 1/z$.

Lévy [Lév] extended Wiener's result as follows: Assume that Φ is holomorphic in a neighborhood of σ. Then $\Phi(f) \in \mathcal{F}$.

How can we generalize this result to arbitrary Banach algebras?

Theorem 3.1. *Let $\mathfrak{A}$ be a Banach algebra and fix $x \in \mathfrak{A}$. Let $\sigma(x)$ denote the spectrum of x. If Φ is any function holomorphic in a neighborhood of $\sigma(x)$, then $\Phi(\hat{x}) \in \hat{\mathfrak{A}}$.*

Note that this contains Lévy's theorem. However, we should like to do better. We want to define an element $\Phi(x) \in \mathfrak{A}$ so as to get a well-behaved map: $\Phi \to \Phi(x)$, not merely to consider the function $\Phi(\hat{x})$ on $\mathcal{M}$. When $\mathfrak{A}$ is not semisimple, this becomes important. We demand that

$$\widehat{\Phi(x)} = \Phi(\hat{x}) \text{ on } \mathcal{M}. \tag{1}$$

The study of a map $\Phi \to \Phi(x)$, from $H(\Omega) \to \mathfrak{A}$, we call the *operational calculus (in one variable)*.

For certian holomorphic functions Φ it is obvious how to define $\Phi(x)$. Let Φ be a polynomial

$$\Phi(z) = \sum_{n=0}^{N} a_n z^n.$$

We put

$$\Phi(x) = \sum_{n=0}^{N} a_n x^n. \tag{2}$$

Note that (1) holds. Let Φ be a rational function holomorphic on $\sigma(x)$,

$$\Phi(z) = \frac{P(z)}{Q(z)},$$

P and Q being polynomials and $Q(z) \neq 0$ for $z \in \sigma(x)$. Then

$$(Q(x))^{-1} \in \mathfrak{A} \qquad \text{(why?)}$$

and we define

$$\Phi(x) = P(x) \cdot Q(x)^{-1} \tag{3}$$

We again verify (1).

Now let Ω be an open set with $\sigma(x) \subset \Omega$ and fix $\Phi \in H(\Omega)$. It follows from Theorem 2.9 that we can choose a sequence $\{f_n\}$ of rational functions holomorphic in Ω such that $f_n \to \Phi$ uniformly on compact subsets of Ω. (Why?) For each n, $f_n(x)$ was defined above. We want to define

$$\Phi(x) = \lim_{n\to\infty} f_n(x).$$

To do this, we must prove

Lemma 3.2. $\lim_{n\to\infty} f_n(x)$ *exist in* $\mathfrak{A}$ *and depends only on* x *and* Φ, *not on the choice of* $\{f_n\}$.

We need

*Exercise 3.2. Let $x \in \mathfrak{A}$, let Ω be an open set containing $\sigma(x)$, and let f be a rational functional holomorphic in Ω.

Choose an open set Ω_1 with

$$\sigma(x) \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega$$

whose boundary γ is the union of finitely many simple closed polygonal curves. Then

$$f(x) = \frac{1}{2\pi i} \int_\gamma f(t) \cdot (t - x)^{-1}\, dt. \tag{4}$$

Proof of Lemma 3.2. Choose γ as in Exercise 3.2. Then

$$\left\| f_n(x) - \frac{1}{2\pi i} \int_\gamma \frac{\Phi(t)\, dt}{t - z} \right\| = \left\| \frac{1}{2\pi i} \int_\gamma \frac{f_n(t) - \Phi(t)}{t - x}\, dt \right\|$$
$$\leq \frac{1}{2\pi} \int_\gamma \left| f_n(t) - \Phi(t) \right| \left\| (t - x)^{-1} \right\| \, dx$$

$\to 0$ as $n \to \infty$, since $\|(t - x)^{-1}\|$ is bounded on γ while $f_n \to \Phi$ uniformly on γ. Thus

$$\lim_{n\to\infty} f_n(x) = \frac{1}{2\pi i} \int_\gamma \frac{\Phi(t)\, dt}{t - x}. \tag{5}$$

□

Now let $\{F_n\}$ be a sequence in $H(\Omega)$. We write

$$F_n \to F \text{ in } H(\Omega)$$

if F_n tends to F uniformly on compact sets in Ω.

Theorem 3.3. *Let $\mathfrak{A}$ be a Banach algebra, $x \in \mathfrak{A}$, and let Ω be an open set containing $\sigma(x)$. Then there exists a map $\tau : H(\Omega) \to \mathfrak{A}$ such that the following holds. We write $F(x)$ for $\tau(F)$:*
(a) *τ is an algebraic homomorphism.*
(b) *If $F_n \to F$ in $H(\Omega)$, then $F_n(x) \to F(x)$ in $\mathfrak{A}$.*
(c) *$\widehat{F(x)} = F(\hat{x})$ for all $F \in H(\Omega)$.*
(d) *If F is the identity function, $F(x) = x$.*
(e) *With γ as earlier, if $F \in H(\Omega)$,*

$$F(x) = \frac{1}{2\pi i} \int_\gamma \frac{F(t)\, dt}{t - x}.$$

Properties (a), (b), *and* (d) *define τ uniquely.*

Note. Theorem 3.1 is contained in this result.

Proof. Fix $F \in H(\Omega)$. Choose a sequence of rational functions $\{f_n\} \in H(\Omega)$ with $f_n \to F$ in $H(\Omega)$. By Lemma 3.2

$$\lim_{n\to\infty} f_n(x) \tag{6}$$

exists in $\mathfrak{A}$. We define this limit to be $F(x)$ and τ to be the map $F \to F(x)$.

τ is evidently a homomorphism when restricted to rational functions. Equation (6) then yields (a). Similarly, (c) holds for rational functions and so by (6) in general. Part (d) follows from (6).

Part (e) coincides with (5). Part (b) comes from (e) by direct computation.

Suppose now that τ' is a map from $H(\Omega)$ to $\mathfrak{A}$ satisfying (a), (b), and (d).

By (a) and (d), τ' and τ agree on rational functions. By (b), then $\tau' = \tau$ on $H(\Omega)$. □

We now consider some consequences of Theorem 3.3 as well as some related questions.

Let $\mathfrak{A}$ be a Banach algebra. By a *nontrivial idempotent e in $\mathfrak{A}$* we mean an element e with $e^2 = e$, e not the zero element or the identity. Suppose that e is such an element. Then $1 - e$ is another. e is not in the radical (why?), so $\hat{e} \not\equiv 0$ on $\mathcal{M}$. Similarly, $\widehat{1-e} \not\equiv 0$, so $\hat{e} \not\equiv 1$. But $\hat{e}^2 = \hat{e}$, so $\hat{e}$ takes on only the values 0 and 1 on $\mathcal{M}$. It follows that $\mathcal{M}$ is disconnected.

QUESTION. Does the converse hold? That is, if $\mathcal{M}$ is disconnected, must $\mathfrak{A}$ contain a nontrivial idempotent?

At this moment, we can prove only a weaker result.

Corollary. *Assume there is an element x in $\mathfrak{A}$ such that $\sigma(x)$ is disconnected. Then $\mathfrak{A}$ contains a nontrivial idempotent.*

PROOF. $\sigma(x) = K_1 \cup K_2$, where K_1, K_2 are disjoint closed sets. Choose disjoint open sets Ω_1 and Ω_2,

$$K_1 \subset \Omega_1, \qquad K_2 \subset \Omega_2.$$

Put $\Omega = \Omega_1 \cup \Omega_2$. Define F on Ω by

$$F = 1 \text{ on } \Omega_1, \qquad F = 0 \text{ on } \Omega_2.$$

Then $F \in H(\Omega)$. Put

$$e = F(x).$$

By Theorem 3.3,

$$e^2 = F^2(x) = F(x) = e$$

and

$$\hat{e} = F(\hat{x}) = \begin{cases} 1 & \text{on } \hat{x}^{-1}(K_1), \\ 0 & \text{on } \hat{x}^{-1}(K_2). \end{cases}$$

Hence e is a nontrivial idempotent. □

EXERCISE 3.3. Let B be a Banach space and T a bounded linear operator on B having disconnected spectrum. Then, there exists a bounded linear operator E on B, $E \neq 0$, $E \neq I$, such that $E^2 = E$ and E commutes with T.

EXERCISE 3.4. Let $\mathfrak{A}$ be a Banach algebra. Assume that $\mathcal{M}$ is a finite set. Then there exist idempotents $e_1, e_2, \ldots, e_n \in \mathfrak{A}$ with $e_i \cdot e_j = 0$ if $i \neq j$ and with $\sum_{i=1}^n e_i = 1$ such that the following holds:

Every x in $\mathfrak{A}$ admits a representation

$$x = \sum_{i=1}^{n} \lambda_i e_i + \rho,$$

where the λ_i are scalars and ρ is in the radical.

Note. Exercise 3.4 contains the following classical fact: If α is an $n \times n$ matrix with complex entries, then there exist commuting matrices β and γ with β nilpotent, γ diagonalizable, and

$$\alpha = \beta + \gamma.$$

To see this, put $\mathfrak{A}$ = algebra of all polynomials in α, normed so as to be a Banach algebra, and apply the exercise.

We consider another problem. Given a Banach algebra $\mathfrak{A}$ and an invertible element $x \in \mathfrak{A}$, when can we find $y \in \mathfrak{A}$ so that

$$x = e^y?$$

There is a purely topological necessary condition: There must exist f in $C(\mathcal{M})$ so that

$$\hat{x} = e^f \text{ on } \mathcal{M}.$$

(Think of an example where this condition is not satisfied.)

We can give a sufficient condition:

Corollary. *Assume that $\sigma(x)$ is contained in a simply connected region Ω, where $0 \notin \Omega$. Then there is a y in $\mathfrak{A}$ with $x = e^y$.*

PROOF. Let Φ be a single-valued branch of $\log z$ defined in Ω. Put $y = \Phi(x)$.

$$\sum_0^N \frac{\Phi^n}{n!} \to e^\Phi = z \text{ in } H(\Omega), \qquad \text{as } N \to \infty.$$

Hence by Theorem 3.3(b),

$$\left(\sum_0^N \frac{\Phi^n}{n!} \right)(x) \to x.$$

By (a) the left side equals

$$\sum_0^N \frac{(\Phi(x))^n}{n!} \to e^y.$$

Hence $e^y = x$. $\square$

To find complete answers to the questions about existence of idempotents and representation of elements as exponentials, we need some more machinery.

We shall develop this machinery, concerning differential forms and the $\bar{\partial}$-operator, in the next three sections. We shall then use the machinery to set up an operational calculus in several variables for Banach algebras, to answer the above questions, and to attack various other problems.

NOTES

Theorem 3.3 has a long history. See E. Hille and R. S. Phillips, Functional analysis and semi-groups, *Am. Math. Soc. Coll. Publ. XXXI*, 1957, Chap. V. In the form given here, it is part of Gelfand's theory [Ge]. For the result on idempotents and related results, see Hille and Phillips, *loc. cit*.

4

Differential Forms

Note. The proofs of all lemmas in this section are left as exercises.

The notion of differential form is defined for arbitrary differentiable manifolds. For our purposes, it will suffice to study differential forms on an open subset Ω of real Euclidean N-space $\mathbb{R}^N$. Fix such an Ω. Denote by $x_1, \ldots, x_N$ the coordinates in $\mathbb{R}^N$.

Definition 4.1. $C^\infty(\Omega) =$ algebra of all infinitely differentiable complex-valued functions on Ω.

We write C^∞ for $C^\infty(\Omega)$.

Definition 4.2. Fix $x \in \Omega$. T_x is the collection of all maps $v : C^\infty \to \mathbb{C}$ for which
(a) v is linear.
(b) $v(f \cdot g) = f(x) \cdot v(g) + g(x) \cdot v(f)$, $f, g \in C^\infty$.

T_x evidently forms a vector space over $\mathbb{C}$. We call it the *tangent space* at x and its elements *tangent vectors* at x.

Denote by $\partial/\partial X_j|_x$ the functional $f \to (\partial f/\partial x_j)(x)$. Then $\partial/\partial x_j|_x$ is a tangent vector at x for $j = 1, 2, \ldots, n$.

Lemma 4.1. $\partial/\partial x_1|_x, \ldots, \partial/\partial x_N|_x$ *forms a basis for* T_x.

Definition 4.3. The dual space to T_x is denoted T_x^*.

Note. The dimension of T_x^* over $\mathbb{C}$ is N.

Definition 4.4. A 1-*form* ω on Ω is a map ω assigning to each x in Ω an element of T_x^*.

EXAMPLE. Let $f \in C^\infty$. For $x \in \Omega$, put

$$(df)_x(v) = v(f), \qquad \text{all } v \in T_x.$$

Then $(df)_x \in T_x^*$.

df is the 1-form on Ω assigning to each x in Ω the element $(df)_x$.

Note. $dx_1, \ldots, dx_N$ are particular 1-forms. In a natural way 1-forms may be added and multiplied by scalar functions.

Lemma 4.2. *Every 1-form ω admits a unique representation*

$$\omega = \sum_1^N C_j dx_j,$$

the C_j being scalar functions on Ω.

Note. For $f \in C^\infty$,

$$df = \sum_{j=1}^N \frac{\partial f}{dx_j} dx_j.$$

We now recall some multilinear algebra. Let V be an N-dimensional vector space over $\mathbb{C}$. Denote by $\wedge^k(V)$ the vector space of k-linear alternating maps of $V \times \cdots \times V \to \mathbb{C}$. ("Alternating" means that the value of the function changes sign if two of the variables are interchanged.)

Define $\mathcal{G}(V)$ as the direct sum

$$\mathcal{G}(V) = \wedge^0(V) \oplus \wedge^1(V) \oplus \cdots \oplus \wedge^N(V).$$

Here $\wedge^0(V) = \mathbb{C}$ and $\wedge^1(V)$ is the dual space of V. Put $\wedge^j(V) = 0$ for $j > N$.

We now introduce a multiplication into the vector space $\mathcal{G}(V)$. Fix $\tau \in \wedge^k(V)$, $\sigma \in \wedge^1(V)$. The map

$$(\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_{k+1}) \to \tau(\xi_1, \ldots, \xi_k)\sigma(\xi_{k+1}, \ldots, \xi_{k+1})$$

is a $(k+l)$-linear map from $V \times \cdots \times V (k+l$ factors$) \to \mathbb{C}$. It is, however, not alternating. To obtain an alternating map, we use

Definition 4.5. Let $\tau \in \wedge^k(V)$, $\sigma \in \wedge^l(V)$, $k, l \geq 1$.

$$\begin{aligned} &\tau \wedge \sigma(\xi_1, \ldots, \xi_{k+1}) \\ &\quad = \frac{1}{(k+l)!} \sum_\pi (-1)^\pi \tau(\xi_{\pi(1)}, \ldots, \xi_{\pi(k)}) \cdot \sigma(\xi_{\pi(k+1)}, \ldots, \xi_{\pi(k+l)}), \end{aligned}$$

the sum being taken over all permutations π of the set $\{1, 2, \ldots, k+l\}$, and $(-1)^\pi$ denoting the sign of the permutation π.

Lemma 4.3. $\tau \wedge \sigma$ *as defined is $(k+l)$-linear and alternating and so* $\in \wedge^{k+l}(V)$.

The operation $\wedge$ (wedge) defines a product for pairs of elements, one in $\wedge^k(V)$ and one in $\wedge^l(V)$, the value lying in $\wedge^{k+l}(V)$, hence in $\mathcal{G}(V)$. By linearity, $\wedge$

extends to a product on arbitrary pairs of elements of $\mathcal{G}(V)$ with value in $\mathcal{G}(V)$. For $\tau \in \wedge^0(V)$, $\sigma \in \mathcal{G}(V)$, define $\tau \wedge \sigma$ as scalar multiplication by τ.

Lemma 4.4. *Under $\wedge$, $\mathcal{G}(V)$ is an associative algebra with identity.*

$\mathcal{G}(V)$ is not commutative. In fact,

Lemma 4.5. *If $\tau \in \wedge^k(V)$, $\sigma \in \wedge^l(V)$, then $\tau \wedge \sigma = (-1)^{kl}\sigma \wedge \tau$.*

Let $e_1, \ldots, e_N$ form a basis for $\wedge^1(V)$.

Lemma 4.6. *Fix k. The set of elements*

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}, \qquad 1 \le i_1 < i_2 < \cdots < i_k \le N,$$

forms a basis for $\wedge^k(V)$.

We now apply the preceding to the case when $V = T_x$, $x \in \Omega$. Then $\wedge^k(T_x)$ is the space of all k-linear alternating functions on T_x, and so, for $k = 1$, coincides with T_x^*. The following thus extends our definition of a 1-form.

Definition 4.6. A *k-form* ω^k on Ω is a map ω^k assigning to each x in Ω an element of $\wedge^k(T_x)$.

k-forms form a module over the algebra of scalar functions on Ω in a natural way.

Let τ^k and σ^l be, respectively, a k-form and an l-form. For $x \in \Omega$, put

$$\tau^k \wedge \sigma^l(x) = \tau^k(x) \wedge \sigma^l(x) \in \wedge^{k+1}(T_x).$$

In particular, since $dx_1, \ldots, dx_N$ are 1-forms,

$$dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$$

is a k-form for each choice of $(i_1, \ldots, i_k)$.

Because of Lemma 4.5,

$$dx_j \wedge dx_j = 0 \text{ for each } j.$$

Hence $dx_{i_1} \wedge \cdots \wedge dx_{i_k} = 0$ unless the i_v are distinct.

Lemma 4.7. *Let ω^k be any k-form on Ω. Then there exist (unique) scalar functions $C_{i_1}, \ldots, i_k$ on Ω such that*

$$\omega^k = \sum_{i_1 < i_2 < \cdots < i_k} C_{i_1} \cdots i_k dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Definition 4.7. $\wedge^k(\Omega)$ consists of all k-forms ω^k such that the functions $C_{i_1} \ldots i_k$ occurring in Lemma 4.7 lie in C^∞. $\wedge^0(\Omega) = C^\infty$.

Recall now the map $f \to df$ from $C^\infty \to \wedge^1(\Omega)$. We wish to extend d to a linear map $\wedge^k(\Omega) \to \wedge^{k+1}(\Omega)$, for all k.

Definition 4.8. Let $\omega^k \in \wedge^k(\Omega)$, $k = 0, 1, 2, \ldots$. Then

$$\omega^k = \sum_{i_1 < \cdots < i_k} C_{i_1} \cdots {}_{i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Define

$$d\omega^k = \sum_{i_1 < \cdots < i_k} dC_{i_1} \cdots {}_{i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Note that d maps $\wedge^k(\Omega) \to \wedge^{k+1}(\Omega)$. We call $d\omega^k$ the *exterior derivative* of ω^k.

For $\omega \in \wedge^1(\Omega)$,

$$\omega = \sum_{i=1}^{N} C_i dx_i,$$

$$d\omega = \sum_{i,j} \frac{\partial C_i}{\partial x_j} dx_j \wedge dx_i = \sum_{i<j} \left(\frac{\partial C_j}{\partial x_i} - \frac{\partial C_i}{\partial x_j} \right) dx_i \wedge dx_j.$$

It follows that for $f \in C^\infty$,

$$\begin{aligned} d(df) &= d\left(\sum_{i=1}^{N} \frac{\partial f}{dx_i} dx_i \right) \\ &= \sum_{i<j} \left(\frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right) - \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_i} \right) \right) dx_i \wedge dx_j = 0 \end{aligned}$$

or $d^2 = 0$ on C^∞. More generally,

Lemma 4.8. $d^2 = 0$ *for every* k*; i.e., if* $\omega^k \in \wedge^k(\Omega)$*,* k *arbitrary, then* $d(d\omega^k) = 0$.

To prove Lemma 4.8, it is useful to prove first

Lemma 4.9. *Let* $\omega^k \in \wedge^k(\Omega)$*,* $\omega^l \in \wedge^l(\Omega)$*. Then*

$$d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l.$$

NOTES

For an exposition of the material in this section, see, e.g., I. M. Singer and J. A. Thorpe, *Lecture Notes on Elementary Topology and Geometry*, Scott, Foresman, Glenview, Ill., 1967, Chap. V.

5

The $\bar{\partial}$-Operator

Note. As in the preceding section, the proofs in this section are left as exercises.

Let Ω be an open subset of $\mathbb{C}^n$.

The complex coordinate functions $z_1, \ldots, z_n$ as well as their conjugates $\bar{z}_1, \ldots, \bar{z}_n$ lie in $C^\infty(\Omega)$. Hence the forms

$$dz_1, \ldots, dz_n, \qquad d\bar{z}_1, \ldots, d\bar{z}_n$$

all belong to $\wedge^1(\Omega)$. Fix $x \in \Omega$. Note that $\wedge^1(T_x) = T_x^*$ has dimension $2n$ over $\mathbb{C}$, since $\mathbb{C}^n = \mathbb{R}^{2n}$. If $x_j = \mathrm{Re}(z_j)$ and $y_j = \mathrm{Im}(z_j)$, then

$$(dx_1)_x, \ldots, (dx_n)_x, \qquad (dy_l)_x, \ldots, (dy_n)_x$$

form a basis for T_x^*. Since $dx_j = 1/2(dz_j + d\bar{z}_j)$ and $dy_j = 1/2i(dz_j - d\bar{z}_j)$,

$$(dz_1)_x, \ldots, (dz_n)_x, \qquad (d\bar{z}_1)_x, \ldots, (d\bar{z}_n)_x$$

also form a basis for T_x^*. In fact,

Lemma 5.1. *If $\omega \in \wedge^1(\Omega)$, then*

$$\omega = \sum_{j=1}^n a_j dz_j + b_j d\bar{z}_j,$$

where $a_j, b_j \in C^x$.

Fix $f \in C^x$. Since $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are real coordinates in $\mathbb{C}^n$,

$$\begin{aligned} df &= \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \frac{\partial f}{\partial y_j} dy_j \\ &= \sum_{j=1}^n \left(\frac{\partial f}{\partial x_j} \cdot \frac{1}{2} + \frac{\partial f}{\partial y_j} \cdot \frac{1}{2i} \right) dz_j + \left(\frac{\partial f}{\partial x_j} \cdot \frac{1}{2} - \frac{1}{2i} \frac{\partial f}{\partial y_j} \right) d\bar{z}_j. \end{aligned}$$

Definition 5.1. We define operators on C^∞ as follows:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

Then for $f \in C^\infty$,

(1) $$df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar z_j} d\bar z_j.$$

Definition 5.2. We define two maps from $C^\infty \to \wedge^1(\Omega)$, ∂ and $\bar\partial$. For $f \in C^\infty$,

$$\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j, \qquad \bar\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar z_j} d\bar z_j.$$

Note. $\partial f + \bar\partial f = df$, if $f \in C^\infty$.

We need some notation. Let I be any r-tuple of integers, $I = (i_1, i_2, \ldots, i_r)$, $1 \le i_j \le n$, all j. Put

$$dz_I = dz_{j_1} \wedge \cdots \wedge dz_{i_r}.$$

Thus $dz_I \in \wedge^r(\Omega)$.

Let J be any s-tuple $(j_1, \ldots, j_s)$, $1 \le j_k \le n$, all k, and put

$$d\bar z_J = d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_s}.$$

So $d\bar z_J \in \wedge^s(\Omega)$. Then

$$dz_I \wedge d\bar z_J \in \wedge^{r+s}(\Omega).$$

For I as above, put $|I| = r$. Then $|J| = s$.

Definition 5.3. Fix integers $r, s \ge 0$. $\wedge^{r,s}(\Omega)$ is the space of all $\omega \in \wedge^{r+s}(\Omega)$ such that

$$\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar z_J,$$

the sum being extended over all I, J with $|I| = r$, $|J| = s$, and with each $a_{IJ} \in C^\infty$.

An element of $\wedge^{r,s}(\Omega)$ is called a *form of type* (r, s). We now have a direct sum decomposition of each $\wedge^k(\Omega)$:

Lemma 5.2.

$$\wedge^k(\Omega) = \wedge^{0,k}(\Omega) \oplus \wedge^{1,k-1}(\Omega) \oplus \wedge^{2,k-2}(\Omega) \oplus \cdots \oplus \wedge^{k,0}(\Omega).$$

We extend the definition of ∂ and $\bar\partial$ (see Definition 5.2) to maps from $\wedge^k(\Omega) \to \wedge^{k+1}(\Omega)$ for k, as follows:

Definition 5.4. Choose ω^k in $\wedge^k(\omega)$,

$$\omega^k = \sum_{I,J} a_{IJ} dz_I \wedge d\bar{z}_J,$$

$$\partial\omega^k = \sum_{I,J} \partial a_{IJ} \wedge dz_I \wedge d\bar{z}_J,$$

and

$$\bar{\partial}\omega^k = \sum_{I,J} \bar{\partial} a_{IJ} \wedge dz_I \wedge d\bar{z}_J.$$

Observe that, by (1), if ω^k is as above,

$$\bar{\partial}\omega^k + \partial\omega^k = \sum_{I,J} da_{IJ} \wedge dz_I \wedge d\bar{z}_J = d\omega^k,$$

so we have

$$\bar{\partial} + \partial = d \tag{2}$$

as operators from $\wedge^k(\Omega) \to \wedge^{k+1}(\Omega)$. Note that if $\omega \in \wedge^{r,s}$, $\partial\omega \in \wedge^{r+1,s}$ and $\bar{\partial}\omega \in \wedge^{r,s+1}$.

Lemma 5.3. $\bar{\partial}^2 = 0$, $\partial^2 = 0$, *and* $\partial\bar{\partial} = \bar{\partial}\partial = 0$.

Why is the $\bar{\partial}$-operator of interest to us? Consider $\bar{\partial}$ as the map from $C^\infty \to \wedge^1(\Omega)$. What is its kernel?

Let $f \in C^\infty$. $\bar{\partial} f = 0$ if and only if

$$\frac{\partial f}{\partial \bar{z}_j} = 0 \text{ in } \Omega, \qquad j = 1, 2, \ldots, n. \tag{3}$$

For $n = 1$ and Ω a domain in the z-plane, (3) reduces to

$$\frac{df}{\partial \bar{z}} = 0 \qquad \text{or} \qquad \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

For $f = u + iv$, u and v real-valued, this means that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

or u and v satisfy the Cauchy-Riemann equations. Thus here

$$\partial f = 0 \text{ in } \Omega \text{ is equivalent to } f \in H(\Omega).$$

Definition 5.5. Let Ω be an open subset of $\mathbb{C}^n$. $H(\Omega)$ is the class of all $f \in C^\infty$ with $\bar{\partial} f = 0$ in Ω, or, equivalently, (3).

We call the elements of $H(\Omega)$ *holomorphic* in Ω. Note that, by (3), $f \in H(\Omega)$ if and only if f is holomorphic in each fixed variable z_j (as the function of a single complex variable), when the remaining variables are held fixed.

Let now Ω be the domain

$$\{z \in \mathbb{C}^n \big| |z_j| < R_j,\ j = 1, \ldots, n\},$$

where $R_1, \ldots, R_n$ are given positive numbers. Thus Ω is a product of n open plane disks. Let f be a once-differentiable function on Ω; i.e., $\partial f/\partial x_j$ and $\partial f/\partial y_j$ exist and are continuous in Ω, $j = 1, \ldots, n$.

Lemma 5.4. *Assume that $\partial f/\partial \bar{z}_j = 0$, $j = 1, \ldots, n$, in Ω. then there exist constants A_ν in $\mathbb{C}$ for each tuple $\nu = (\nu_1, \ldots, \nu_n)$ of nonnegative integers such that*

$$f(z) = \sum_\nu A_\nu z^\nu,$$

where $z^\nu = z_1^{\nu_1} \cdot z_2^{\nu_2} \cdots z_n^{\nu_n}$, the series converging absolutely in Ω and uniformly on every compact subset of Ω.

For a proof of this result, see, e.g., [Hö, Th. 2.2.6].

This result then applies in particular to every f in $H(\Omega)$. We call $\sum_\nu A_\nu z^\nu$ the *Taylor series* for f at 0.

We shall see that the study of the $\bar{\partial}$-operator, to be undertaken in the next section and in later sections, will throw light on the holomorphic functions of several complex variables.

For further use, note also

Lemma 5.5. *If $\omega^k \in \wedge^k(\Omega)$ and $\omega^l \in \wedge^l(\Omega)$, then*

$$\bar{\partial}(\omega^k \wedge \omega^l) = \bar{\partial}\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge \bar{\partial}\omega^l.$$

6

The Equation $\bar{\partial} u = f$

As before, fix an open set $\Omega \subset \mathbb{C}^n$. Given $f \in \wedge^{r,s+1}(\Omega)$, we seek $u \in \wedge^{r,s}$ such that

$$\bar{\partial} u = f. \tag{1}$$

Since $\bar{\partial}^2 = 0$ (Lemma 5.3), a necessary condition on f is

$$\bar{\partial} f = 0. \tag{2}$$

If (2) holds, we say that f is $\bar{\partial}$-*closed*. What is a sufficient condition on f? It turns out that this will depend on the domain Ω.

Recall the analogous problem for the operator d on a domain $\Omega \subset \mathbb{R}^n$. If ω^k is a k-form in $\wedge^k(\Omega)$, the condition

$$d\omega^k = 0 \qquad (\omega \text{ is "closed"}) \tag{3}$$

is necessary in order that we can find some τ^{k-1} in $\wedge^{k-1}(\Omega)$ with

$$d\tau^{k-1} = \omega^k. \tag{4}$$

However, (3) is, in general, not sufficient. (Think of an example when $k = 1$ and Ω is an annulus in $\mathbb{R}^2$.) If Ω is contractible, then (3) is sufficient in order that (4) admit a solution.

For the $\bar{\partial}$-operator, a purely topological condition on Ω is inadequate. We shall find various conditions in order that (1) will have a solution. Denote by Δ^n the closed unit polydisk in $\mathbb{C}^n : \Delta^n = \{z \in \mathbb{C}^n \big| |z_j| \leq 1, \ j = 1, \dots, n\}$.

Theorem 6.1 (Complex Poincaré Lemma). *Let Ω be a neighborhood of Δ^n. Fix $\omega \in \wedge^{p,q}(\Omega)$, $q > 0$, with $\bar{\partial}\omega = 0$. Then there exists a neighborhood Ω^* of Δ^n and there exists $\omega^* \in \wedge^{p,q-1}(\Omega^*)$ such that*

$$\bar{\partial}\omega^* = \omega \textit{ in } \Omega^*.$$

We need some preliminary work.

Lemma 6.2. *Let $\phi \in C^1(\mathbb{R}^2)$ and assume that ϕ has compact support. Put*

$$\Phi(\zeta) = -\frac{1}{\pi} \int_{\mathbb{R}^2} \phi(z) \frac{dx\,dy}{z - \zeta}.$$

Then $\Phi \in C^1(\mathbb{R}^2)$ and $\partial\Phi/\partial\bar{\zeta} = \phi(\zeta)$, all ζ.

PROOF. Choose R with $\operatorname{supp}\phi \subset \{z \big| |z| \leq R\}$.

$$\begin{aligned}\pi\Phi(\zeta) &= \int_{|z|\leq R} \phi(z) \frac{1}{\zeta - z}\, dx\,dy = \int_{|z'-\zeta|\leq R} \phi(\zeta - z') \frac{dx'\,dy'}{z'}\\ &= \int_{\mathbb{R}^2} \phi(\zeta - z') \frac{dx'\,dy'}{z'}.\end{aligned}$$

Since $1/z' \in L^1(dx'\,dy')$ on compact sets, it is legal to differentiate the last integral under the integral sign. We get

$$\begin{aligned}\pi \frac{\partial\Phi}{\partial\bar{\zeta}}(\zeta) &= \int_{\mathbb{R}^2} \frac{\partial}{\partial\bar{\zeta}}[\phi(\zeta - z')] \frac{dx'\,dy'}{z'} = \int_{\mathbb{R}^2} \frac{\partial\phi}{\partial\bar{z}}(\zeta - z') \frac{dx'\,dy'}{z'}\\ &= \int_{\mathbb{R}^2} \frac{\partial\phi}{\partial\bar{z}}(z) \frac{dx\,dy}{\zeta - z}.\end{aligned}$$

On the other hand, Lemma 2.5 gives that

$$-\pi\phi(\zeta) = \int_{\mathbb{R}^2} \frac{\partial\phi}{\partial\bar{z}}(z) \frac{dx\,dy}{z - \zeta}.$$

Hence $\partial\Phi/\partial\zeta = \phi$. □

Lemma 6.3. *Let Ω be a neighborhood of Δ^n and fix f in $C^\infty(\Omega)$. Fix j, $1 \leq j \leq n$. Assume that*

(5) $$\frac{\partial f}{\partial\bar{z}_k} = 0 \mathit{in}\ \Omega,\ k = k_1, \ldots, k_s,\ \mathit{each}\ k_i \neq j.$$

Then we can find a neighborhood Ω_1 of Δ^n and F in $C^\infty(\Omega_1)$ such that
(a) $\partial F/\partial\bar{\zeta}_j = f$ *in* Ω_1.
(b) $\partial F/\partial\bar{\zeta}_k = 0$ *in* Ω_1, $k = k_1, \ldots, k_s$.

PROOF. Choose $\varepsilon > 0$ so that if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $|z_\nu| < 1 + 2\varepsilon$ for all ν, then $z \in \Omega$.

Choose $\psi \in C^\infty(\mathbb{R}^2)$, having support contained in $\{z \big| |z| < 1 + 2\varepsilon\}$, with $\psi(z) = 1$ for $|z| < 1 + \varepsilon$. Put

$$\begin{aligned}&F(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n)\\ &\quad = -\frac{1}{\pi} \int_{\mathbb{R}^2} \psi(z) f(\zeta_1, \ldots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n) \frac{dx\,dy}{z - \zeta_j}.\end{aligned}$$

For fixed $\zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots, \zeta_n$ with $|\zeta_\nu| < 1 + \varepsilon$, all ν, we now apply Lemma 6.2 with

$$\begin{aligned}\phi(z) &= \psi(z) f(\zeta_1, \ldots, \zeta_{j-1}, z, \zeta_{j+1}, \ldots, \zeta_n),\ |z| < 1 + 2\varepsilon\\ &= 0 \qquad \text{outside } \operatorname{supp}\psi.\end{aligned}$$

We obtain

$$\frac{\partial F}{\partial \bar{\zeta}_j}(\zeta_1, \ldots, \zeta_j, \ldots, \zeta_n) = \phi(\zeta_j) = f(\zeta_1, \ldots, \zeta_{j-1}, \zeta_j, \zeta_{j+1}, \ldots, \zeta_n),$$

if $|\zeta_j| < 1 + \varepsilon$, and so (a) holds with

$$\Omega_1 = \{\zeta \in \mathbb{C}^n \mid |\zeta_\nu| < 1 + \varepsilon, \text{ all } \nu\}.$$

$\square$

Part (b) now follows directly from (5) by differentiation under the integral sign.

Proof of Theorem 6.1. We call a form

$$\sum_{I,J} C_{IJ} dz_I \wedge d\bar{z}_J$$

of *level* ν, if for some I and J with $J = (j_1, j_2, \ldots, \nu)$, where $j_1 < j_2 < \cdots < \nu$, we have $C_{IJ} \neq 0$; while for each I and J with $J = (j_1, \ldots, j_s)$ where $j_1 < \cdots < j_s$ and $j_s > \nu$, we have $C_{IJ} = 0$.

Consider first a form ω of level 1 such that $\bar{\partial}\omega = 0$. Then $\omega \in \wedge^{p,1}(\Omega)$ for some p and we have

$$\omega = \sum_I a_I d\bar{z}_1 \wedge dz_I, a_I \in C^\infty(\Omega) \qquad \text{for each } I.$$

$$0 = \bar{\partial}\omega = \sum_{I,k} \frac{\partial a_I}{\partial \bar{z}_k} d\bar{z}_k \wedge d\bar{z}_I \wedge dz_I.$$

Hence $(\partial a_I / \partial \bar{z}_k) d\bar{z}_k \wedge d\bar{z}_1 \wedge dz_I = 0$ for each k and I. It follows that

$$\frac{\partial a_I}{\partial \bar{z}_k} = 0, \qquad k \geq 2, \text{ all } I.$$

By Lemma 6.3 there exists for every I, A_I in $C^\infty(\Omega_1)$, Ω_1 being some neighborhood of Δ^n, such that

$$\frac{\partial A_I}{\partial \bar{z}_1} = a_I \text{ and } \frac{\partial A_I}{\partial \bar{z}_k} = 0, \qquad k = 2, \ldots, n.$$

Put $\tilde{\omega} = \sum_I A_I dz_I \in \wedge^{p,0}(\Omega_1)$.

$$\bar{\partial}\tilde{\omega} = \sum_{I,k} \frac{\partial A_I}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_I = \omega.$$

We proceed by induction. Assume that the assertion of the theorem holds whenever ω is of level $\leq \nu - 1$ and consider ω of level ν. By hypothesis $\omega \in \wedge^{p,q}(\Omega)$ and $\bar{\partial}\omega = 0$.

We can find forms α and β of level $\leq \nu - 1$ so that

$$\omega = d\bar{z}_\nu \wedge \alpha + \beta \qquad \text{(why?)}.$$

$$0 = \bar{\partial}\omega = -d\bar{z}_\nu \wedge \bar{\partial}\alpha + \bar{\partial}\beta,$$

where we have used Lemma 5.5. So

$$0 = d\bar z_\nu \wedge \bar\partial\alpha - \bar\partial\beta. \tag{6}$$

Put

$$\alpha = \sum_{I,J} a_{IJ} dz_I \wedge d\bar z_j, \qquad \beta = \sum_{I,J} b_{IJ} dz_I \wedge d\bar z_J.$$

Equation (6) gives

$$\begin{aligned} 0 = d\bar z_\nu \wedge \sum_{I,J,k} \frac{\partial a_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J \\ - \sum_{I,J,k} \frac{\partial b_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J. \end{aligned} \tag{7}$$

Fix $k > \nu$, and look at the terms on the right side of (7) containing $d\bar z_\nu \wedge d\bar z_k$. Because α and β are the level $\le \nu - 1$, these are the terms:

$$d\bar z_\nu \wedge \frac{\partial a_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J.$$

It follows that for each I and J,

$$\frac{\partial a_{IJ}}{\partial \bar z_k} = 0, \qquad k > \nu.$$

By Lemma 6.3 there exists a neighborhood Ω_1 of Δ^n and, for each I and J, $A_{IJ} \in C^\infty(\Omega_1)$ with

$$\frac{\partial A_{IJ}}{\partial \bar z_\nu} = a_{IJ}, \qquad \frac{\partial A_{IJ}}{\partial \bar z_k} = 0, \qquad k > \nu.$$

Put

$$\begin{aligned} \omega_1 &= \sum_{I,J} A_{IJ} dz_I \wedge d\bar z_J \in \wedge^{p,q-1}(\Omega_1), \\ \bar\partial\omega_1 &= \sum_{I,J,k} \frac{\partial A_{IJ}}{\partial \bar z_k} d\bar z_k \wedge dz_I \wedge d\bar z_J \\ &= \sum_{I,J} a_{IJ} d\bar z_\nu \wedge dz_I \wedge d\bar z_J + \gamma, \end{aligned}$$

where γ is a form of level $\le \nu - 1$. Thus

$$\bar\partial\omega_1 = d\bar z_\nu \wedge \alpha + \gamma.$$

Hence

$$\bar\partial\omega_1 - \omega = \gamma - \beta$$

is a form of level $\le \nu - 1$. Also

$$\bar\partial(\gamma - \beta) = \bar\partial(\bar\partial\omega_1 - \omega) = 0.$$

By induction hypothesis, we can choose a neighborhood Ω_2 of Δ^n and $\tau \in \wedge^{p,q-1}(\Omega_2)$ with $\bar{\partial}\tau = \gamma - \beta$. Then

$$\bar{\partial}(\omega_1 - \tau) = \bar{\partial}\omega_1 - \bar{\partial}\tau = \omega + (\gamma - \beta) - (\gamma - \beta) = \omega.$$

$\omega_1 - \tau$ is now the desired ω^*. □

NOTES

Theorem 6.1 is in P. Dolbeaut, Formes différentielles et cohomologie sur une variété analytique complexe, I, *Ann. Math.* **64** (1956), 83-130; II, *Ann. Math.* **65** (1957), 282-330. For the proof cf. [Hö2, Chap. 2].

7

The Oka-Weil Theorem

Let K be a compact set in the z-plane and denote by $P(K)$ the uniform closure on K of the polynomials in z.

Theorem 7.1. *Assume that $\mathbb{C}\setminus K$ is connected. Let F be holomorphic in some neighborhood Ω of K. Then $F|_K$ is in $P(K)$.*

PROOF. Let $\mathcal{L}$ denote the space of all finite linear combinations of functions $1/(z-a)^p$, where $a \in \mathbb{C}\setminus\Omega$, p an integer ≥ 0. By Runge's theorem (Theorem 2.9), $F|_K$ lies in the uniform closure of $\mathcal{L}$ on K. We claim that $\mathcal{L} \subset P(K)$. For let μ be a measure on K, $\mu \perp P(K)$. Then for $|a|$ large,

$$\int \frac{d\mu(z)}{z-a} = -\int \left(\sum_0^\infty \frac{z^n}{a^{n+1}} \right) d\mu = 0.$$

But the integral on the left is analytic as a function of a in $\mathbb{C}\setminus K$ and, since $\mathbb{C}\setminus K$ is connected, vanishes for all a in $\mathbb{C}\setminus K$. By differentiation,

$$\int \frac{d\mu(z)}{(z-a)^p} = 0, \qquad p = 1, 2, \ldots, a \in \mathbb{C}\setminus K.$$

Thus $\mu \perp \mathcal{L}$, so $\mathcal{L} \subset P(K)$, as claimed. The theorem follows. $\square$

How can we generalize this result to the case when K is a compact subset of $\mathbb{C}^n$, $n > 1$?

What condition on K will assure the possibility of approximating arbitrary functions holomorphic in a neighborhood of K uniformly on K by polynomials in $z_1, \ldots, z_n$?

Note that the condition "$\mathbb{C}\setminus K$ is connected" is a purely topological restriction on K. No such purely topological restriction can suffice when $n > 1$. As an example, consider the two sets in $\mathbb{C}^2$.

$$K_1 = \{(e^{i\theta}, 0) | 0 \leq \theta \leq 2\pi\},$$
$$K_2 = \{(e^{i\theta}, e^{-i\theta}) | 0 \leq \theta \leq 2\pi\}.$$

The two sets are, topologically, circles. The function $F(z_1, z_2) = 1/z_1$ is holomorphic in a neighborhood of K_1.

Yet we cannot approximate F uniformly on K_1 by polynomials in z_1, z_2. (Why?) On the other hand, every continuous function on K_2 is uniformly approximable by polynomials in z_1, z_2. (Why?)

To obtain a general condition valid in $\mathbb{C}^n$ for all n we rephrase the statement "$\mathbb{C}\backslash K$ is connected" as follows:

Lemma 7.2. *Let K be a compact set in $\mathbb{C}$. $\mathbb{C}\backslash K$ is connected if and only if for each $x_0 \in \mathbb{C}\backslash K$ we can find a polynomial P such that*

$$|P(x^0)| > \max_K |P|. \tag{1}$$

PROOF. If $\mathbb{C}\backslash K$ fails to be connected, we can choose x^0 in a bounded component of $\mathbb{C}\backslash K$ and note that (1) violates the maximum principle.

Assume that $\mathbb{C}\backslash K$ is connected. Fix $x^0 \in \mathbb{C}\backslash K$. Then $K \cup \{x^0\}$ is a set with connected complement. Choose points $x_n \to x^0$ and $x_n \neq x^0$. Then

$$f_n(z) = \frac{1}{z - x_n}$$

is holomorphic in a neighborhood of $K \cup \{x^0\}$. Hence by Theorem 7.1 we can find a polynomial P_n with

$$\left| P_n(z) - \frac{1}{z - x_n} \right| < \frac{1}{n}, \qquad \text{all } z \in K \cup \{x_0\}.$$

For large n, then, P_n satisfies (1). $\square$

Definition 7.1. Let X be a compact subset of $\mathbb{C}^n$. We define the *polynomially convex hull of* X, denoted $h(X)$, by

$$h(X) = \{z \in \mathbb{C}^n \big| |Q(z)| \leq \max_X |Q|$$

for every polynomial $Q\}$.

Evidently $h(X)$ is a compact set containing X.

Definition 7.2. X is said to be *polynomially convex* if $h(X) = X$.

Note that X is polynomially convex if and only if for every x^0 in $\mathbb{C}^n\backslash X$ we can find a polynomial P with

$$|P(x^0)| > \max_X |P|. \tag{2}$$

For $X \subset \mathbb{C}$, Lemma 7.2 now gives that $\mathbb{C}\backslash X$ is connected if and only if X is polynomially convex. Theorem 7.1 can now be stated: For $X \subset \mathbb{C}$, the approximation problem on X is solvable provided that X is polynomially convex. Formulated in this way, the theorem admits generalization to $\mathbb{C}^n$ for $n > 1$.

Theorem 7.3 (Oka-Weil). *Let X be a compact, polynomially convex set in $\mathbb{C}^n$. Then for every function f holomorphic in some neighborhood of X, we can find a* *sequence* $\{P_j\}$ *of polynomials in* $z_1, \ldots, z_n$ *with*

$$P_j \to f \textit{ uniformly on } X.$$

Note. In order to apply this result in particular cases we of course have to verify that a given set X is polynomially convex. This is usually quite difficult. However, we shall see that in the theory of Banach algebras polynomially convex sets arise in a natural way.

André Weil, who first proved the essential portion of Theorem 7.3 [L'Intégrale de Cauchy et les fonctions de plusieurs variables, *Math. Ann.* **111** (1935), 178-182], made use of a generalization of the Cauchy integral formula to several complex variables. We shall follow another route, due to Oka, based on the Oka extension theorem given below.

Definition 7.3. A subset Π of $\mathbb{C}^n$ is a *p-polyhedron* if there exist polynomials $P_1, \ldots, P_s$ such that

$$\Pi = \{z \in \mathbb{C}^n \big| |z_j| \leq 1, \text{ all } j, \text{ and } |P_k(z)| \leq 1, k = 1, 2, \ldots, s\}.$$

Lemma 7.4. *Let* X *be a compact polynomially convex subset of* Δ^n *Let* $\mathcal{O}$ *be an open set containing* X. *Then there exists a p-polyhedron* Π *with* $X \subset \Pi \subset \mathcal{O}$.

Proof. For each $x \in \Delta^n \backslash \mathcal{O}$ there exists a polynomial P_x with $|P_x(x)| > 1$ and $|P_x| \leq 1$ on X.

Then $|P_x| > 1$ in some neighborhood $\mathcal{N}_x$ of x. By compactness of $\Delta^n \backslash \mathcal{O}$, a finite collection $\mathcal{N}_{x_1}, \ldots, \mathcal{N}_{x_r}$ covers $\Delta^n \backslash \mathcal{O}$. Put

$$\Pi = \{z \in \Delta^n \big| |P_{x_1}(z)| \leq 1, \ldots, |P_{x_r}(z)| \leq 1\}.$$

If $z \in X$, then $z \in \Pi$, so $X \subset \Pi$.

Suppose that $z \notin \mathcal{O}$. If $z \notin \Delta^n$, then $z \notin \Pi$. If $z \in \Delta^n$, then $z \in \Delta^n \backslash \mathcal{O}$. Hence $z \in \mathcal{N}_{xj}$ for some j. Hence $|P_{xj}(z)| > 1$. Thus $z \notin \Pi$. Hence $\Pi \subset \mathcal{O}$. □

Let now Π be a p-polyhedron in $\mathbb{C}^n$,

$$\Pi = \{z \in \Delta^n \big| |P_j(z)| \leq 1, j = 1, \ldots, r\}.$$

We can embed Π in $\mathbb{C}^{n+r}$ by the map

$$\Phi : z \to (z, P_1(z), \ldots, P_r(z)).$$

Φ maps Π homeomorphically onto the subset of Δ^{n+r} defined by the equations

$$z_{n+1} - P_1(z) = 0, \ldots, z_{n+r} - P_r(z) = 0.$$

Theorem 7.5 (Oka Extension Theorem). *Given* f *holomorphic in some neighborhood of* Π*; then there exists* F *holomorphic in a neighborhood of* Δ^{n+r} *such that*

$$F(z, P_1(z), \ldots, P_r(z)) = f(z), \textit{ all } z \in \Pi.$$

The Oka-Weil theorem is an easy corollary of this result.

PROOF OF THEOREM 7.3. Without loss of generality we may assume that $X \subset \Delta^n$. (Why?) f is holomorphic in a neighborhood $\mathcal{O}$ of X. By Lemma 7.4 there exists a p-polyhedron Π with $X \subset \Pi \subset \mathcal{O}$. Then f is holomorphic in a neighborhood of Π. By Theorem 7.5 we can find F satisfying

$$F(z, P_1(z), \ldots, P_r(z)) = f(z), \qquad z \in \Pi, \tag{3}$$

F holomorphic in a neighborhood of Δ^{n+r}. Expand F in a Taylor series around 0,

$$F(z, z_{n+1}, \ldots, z_{n+r}) = \sum_\nu a_\nu z_1^{\nu_1} \cdots z_n^{\nu_n} z_{n+1}^{\nu_{n+1}} \cdots z_{n+r}^{\nu_{n+r}}.$$

The series converges uniformly in Δ^{n+r}. Thus a sequence $\{S_j\}$ of partial sums of this series converges uniformly to F on Δ^{n+r}, and hence in particular on $\Phi(\Pi)$. Thus

$$S_j(z, P_1(z), \ldots, P_r(z))$$

converges uniformly to $F(z, P_1(z), \ldots, P_r(z))$ for $z \in \Pi$, or, in other words, converges to $f(z)$, by (3). Since $S_j(z, P_1(z), \ldots, P_r(z))$ is a polynomial in z for each j, we are done.

We must now attack the Oka Extension theorem. We begin with a generalization of Theorem 6.1.

Theorem 7.6. *Let Π be a p-polyhedron in $\mathbb{C}^n$ and Ω a neighborhood of Π. Given that $\phi \in \wedge^{p,q}(\Omega)$, $q > 0$, with $\bar\partial\phi = 0$, then there exists a neighborhood Ω_1 of Π and $\psi \in \wedge^{p,q-1}(\Omega_1)$ with $\bar\partial\psi = \phi$.*

First we need some definitions and exercises.

Let Ω be an open set in $\mathbb{C}^n$ and W and open set in $\mathbb{C}^k$. Let $u = (u_1, \ldots, u_n)$ be a map of W into Ω. Assume that each $u_j \in C^\infty(W)$.

EXERCISE 7.1. Let $a \in C^\infty(\Omega)$, so $a(u) \in C^\infty(W)$. Then

$$d\{a(u)\} = \sum_{j=1}^n \frac{\partial a}{\partial z_j}(u)du_j + \frac{\partial a}{\partial \bar z_j}(u)d\bar u_j.$$

Both sides are forms in $\wedge^1(W)$.

Let Ω, W, and u be as above. Assume that each $u_j \in H(W)$. For each $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_s)$ put

$$du_I = du_{i_1} \wedge du_{i_2} \wedge \cdots \wedge du_{i_r}$$

and define $d\bar u_J$ similarly. Thus $du_I \wedge d\bar u_J \in \wedge^{r,s}(W)$.

Fix $\omega \in \wedge^{r,s}(\Omega)$,

$$\omega = \sum_{I,J} a_{IJ} dz_I \wedge d\bar z_J.$$

Definition 7.4.

$$\omega(u) = \sum_{I,J} a_{IJ}(u) du_I \wedge d\bar{u}_J \in \wedge^{r,s}(W).$$

EXERCISE 7.2. $d(\omega(u)) = (d\omega)(u)$ and $\bar{\partial}(\omega(u)) = (\bar{\partial}\omega)(u)$. We still assume, in this exercise, that each u_j is holomorphic.

PROOF OF THEOREM 7.6. We denote

$$P^k(q_1, \ldots, q_r) = \{z \in \Delta^k \big| |q_j(z)| \leq 1,\ j = 1, \ldots, r\},$$

the q_j being polynomials in $z_1, \ldots, z_k$. Every p-polyhedron is of this form.

We shall prove our theorem by induction on r. The case $r = 0$ corresponds to the p-polyhedron Δ^k and the assertion holds, for all k, by Theorem 6.1.

Fix r now and suppose that the assertion holds for this r and all k and all $(p, q), q > 0$. Fix n and polynomials $p_1, \ldots, p_{r+1}$ in $\mathbb{C}^n$ and consider $\phi \in \wedge^{p,q}(\Omega)$, Ω some neighborhood of $P^n(p_1, \ldots, p_{r+1})$. We first sketch the argument.

Step 1. Embed $P^n(p_1, \ldots, p_{r+1})$ in $P^{n+1}(p_1, \ldots, p_r)$ by the map $u : z \to (z, p_{r+1}(z))$. Note that $p_1, \ldots, p_r$ are polynomials in $z_1, \ldots, z_{n+1}$ which do not involve z_{n+1}. Let $\sum$ denote the image of $P^n(p_1, \ldots, p_{r+1})$ under u. π denotes the projection $(z, z_{n+1}) \to z$ from $\mathbb{C}^{n+1} \to \mathbb{C}^n$. Note $\pi \circ u =$ identity.

Step 2. Find a $\bar{\partial}$-closed form Φ_1 defined in a neighborhood of

$$P^{n+1}(p_1, \ldots, p_r)$$

with $\Phi_1 = \phi(\pi)$ on $\sum$.

Step 3. By induction hypothesis, $\exists\Psi$ in a neighborhood of $P^{n+1}(p_1, \ldots, p_r)$ with $\bar{\partial}\Psi = \Phi_1$. Put $\psi = \Psi(u)$. Then

$$\bar{\partial}\psi = (\bar{\partial}\Psi)(u) = \Phi_1(u) = \phi.$$

As to the details, choose a neighborhood Ω_1 of $P^n(p_1, \ldots, p_{r+1})$ with $\bar{\Omega}_1 \subset \Omega$. Choose $\lambda \in C^\infty(\mathbb{C}^n)$, $\lambda = 1$ on $\bar{\Omega}_1$, $\lambda = 0$ outside Ω. Put $\Phi = (\lambda \cdot \phi)(\pi)$, defined $= 0$ outside $\pi^{-1}(\Omega)$.

Let χ be a form of type (p, q) defined in a neighborhood of $P^{n+1}(p_1, \ldots, p_r)$. Put

$$\Phi_1 = \Phi - (z_{n+1} - p_{r+1}(z)) \cdot \chi. \tag{4}$$

Then $\Phi_1 = \Phi = \phi(\pi)$ on $\sum$.

We want to choose χ such that Φ_1 is $\bar{\partial}$-closed. This means that

$$\bar{\partial}\Phi = (z_{n+1} - p_{r+1}(z))\bar{\partial}\chi$$

or

$$\bar{\partial}\chi = \frac{\bar{\partial}\Phi}{(z_{n+1} - p_{r+1}(z))}. \tag{5}$$

Observe that $\bar{\partial}\Phi = \bar{\partial}\phi(\pi) = 0$ in a neighborhood of $\sum$, whence the right-hand side in (5) can be taken to be 0 in a neighborhood of $\sum$ and is then in C^∞ in a neighborhood of $P^{n+1}(p_1, \ldots, p_r)$. Also

$$\bar{\partial}\left\{\frac{\bar{\partial}\Phi}{(z_{n+1} - p_{r+1}(z))}\right\} = 0.$$

By induction hypothesis, now, $\exists\chi$ satisfying (5). The corresponding Φ_1 in (4) is then $\bar{\partial}$-closed in some neighborhood of $P^{n+1}(p_1, \ldots, p_r)$. By induction hypothesis again, $\exists a$ $(p, q-1)$ form Ψ in a neighborhood of $P^{n+1}(p_1, \ldots, p_r)$ with $\bar{\partial}\Psi = \Phi_1$. As in step 3, then, making use of Exercise 7.2, we obtain a $(p, q-1)$ form ψ in a neighborhood of $P^n(p_1, \ldots, p_{r+1})$ with $\bar{\partial}\psi = \phi$. □

We keep the notations introduced in the last proof.

Lemma 7.7. *Fix k and polynomials $q_1, \ldots, q_r$ in $z = (z_1, \ldots, z_k)$. Let f be holomorphic in a neighborhood W of $\Pi = P^k(q_1, \ldots, q_r)$. The $\exists F$ holomorphic in a neighborhood of $\Pi' = P^{k+1}(q_2, \ldots, q_r)$ such that*

$$F(z, q_1(z)) = f(z), \qquad \textit{all } z \in \Pi.$$

[Note that if $z \in \Pi$, then $(z, q_1(z)) \in \Pi'$.]

Proof. Let $\sum$ be the subset of Π' defined by $z_{k+1} - q_1(z) = 0$. Choose $\phi \in C_0^\infty(\pi^{-1}(W))$ with $\phi = 1$ in a neighborhood of $\sum$.

We seek a function G defined in a neighborhood of Π' so that with

$$F(z, z_{k+1}) = \phi(z, z_{k+1})f(z) - (z_{k+1} - q_1(z))G(z, z_{k+1}),$$

F is holomorphic in a neighborhood of Π'. We define $\phi \cdot f = 0$ outside $\pi^{-1}(W)$. We need $\bar{\partial}F = 0$ and so

$$f\bar{\partial}\phi = (z_{k+1} - q_1(z))\bar{\partial}G$$

or

$$\bar{\partial}G = \frac{f\bar{\partial}\phi}{(z_{k+1} - q_1(z))} = \omega. \tag{6}$$

Note that the numerator vanishes in a neighborhood of $\sum$, so ω is a smooth form in some neighborhood of Π'. Also $\bar{\partial}\omega = 0$. By Theorem 7.6, we can thus find G satisfying (6) in some neighborhood or Π'. The corresponding F now has the required properties. □

Proof of Theorem 7.5. $p_1, \ldots, p_r$ are given polynomials in $z_1, \ldots, z_n$ and $\Pi = P^n(p_1, \ldots, p_r)$. f is holomorphic in a neighborhood of Π. For $j = 1, 2, \ldots, r$

we consider the assertion

$$A(j) : \exists F_j \text{holomorphic in a neighborhood of } P^{n+j}(p_{j+1}, \ldots, p_r)$$

such that $F_j(z, p_1(z), \ldots, p_j(z)) = f(z)$, all $z \in \Pi$.

$A(1)$ holds by Lemma 7.7. Assume that $A(j)$ holds for some j. Thus F_j is holomorphic in a neighborhood of $P^{n+j}(p_{j+1}, \ldots, p_r)$. By Lemma 7.7, $\exists F_{j+1}$ is holomorphic in a neighborhood of $P^{n+j+1}(p_{j+2}, \ldots, p_r)$ with $F_{j+1}(\zeta, p_{j+1}(z)) = F_j(\zeta)$, $\zeta \in P^{n+j}(p_{j+1}, \ldots, p_r)$, and $\zeta = (z, z_{n+1}, \ldots, z_{n+j})$.

By choice of F_j.

$$F_j(z, p_1(z), \ldots, p_j(z)) = f(z), \qquad \text{all } z \text{ in } \Pi.$$

Hence

$$F_{j+1}(z, p_1(z), \ldots, p_j(z), p_{j+1}(z)) = f(z), \qquad \text{all } z \text{ in } \Pi.$$

Thus $A(j+1)$ holds. Hence $A(1), A(2), \ldots, A(r)$ all hold. But $A(r)$ provides F holomorphic in a neighborhood of Δ^{n+r} with

$$F(z, p_1(z), \ldots, p_r(z)) = f(z), \qquad \text{all } z \text{ in } \Pi.$$

□

EXERCISE 7.3. Let $\mathfrak{A}$ be a uniform algebra on a compact space X with generators $g_1, \ldots, g_n$ (i.e., $\mathfrak{A}$ is the smallest closed subalgebra of itself containing the g_j). Show that the map

$$x \to (\hat{g}_1(x), \ldots, \hat{g}_n(x))$$

maps $\mathcal{M}(\mathfrak{A})$ onto a compact, polynomially convex set K in $\mathbb{C}^n$, and that this map carries $\mathfrak{A}$ isomorphically and isometrically onto $P(K)$.

EXERCISE 7.4. Let X be a compact set in $\mathbb{C}^n$. Show that $\mathcal{M}(P(X))$ can be identified with $h(X)$. In particular, if X is polynomially convex, $\mathcal{M}(P(X)) = X$.

NOTES

Theorem 7.5 and the proof of Theorem 7.3 based on it is due to K. Oka, Domaines convexes par rapport aux fonctions rationelles, *J. Sci. Hiroshima Univ.* **6** (1936), 245-255. The proof of Theorem 7.5 given here is found in Gunning and Rossi [GR, Chap. 1].

8

Operational Calculus in Several Variables

We wish to extend the operational calculus established in Section 3 to functions of several variables. Let $\mathfrak{A}$ be a Banach algebra and $x_1, \ldots, x_n \in \mathfrak{A}$. If P is a polynomial in n variables

$$P(z_1, \ldots, z_n) = \sum_\nu A_\nu z_1^{\nu_1} \cdots z_n^{\nu_n},$$

it is natural to define

$$P(x_1, \ldots, x_n) = \sum_\nu A_\nu x_1^{\nu_1} \cdots x_n^{\nu_n} \in \mathfrak{A}.$$

We then observe that if $y = P(x_1, x_2, \ldots, x_n)$, then

(1) $$\hat{y} = P(\hat{x}_1, \ldots, \hat{x}_n) \text{ on } \mathcal{M}.$$

Let F be a complex-valued function defined on an open set $\Omega \subset \mathbb{C}^n$. In order to define $F(\hat{x}_1, \ldots, \hat{x}_n)$ on $\mathcal{M}$ we must assume that Ω contains

$$\{(\hat{x}_1(M), \ldots, \hat{x}_n(M)) | M \in \mathcal{M}\}.$$

Definition 8.1. $\sigma(x_1, \ldots, x_n)$, *the joint spectrum of* $x_1, \ldots, x_n$, *is* $\{(\hat{x}_1(M), \ldots, \hat{x}_n(M)) | M \in \mathcal{M}\}$.

When $n = 1$, we recover the old spectrum $\sigma(x)$. You easily verify

Lemma 8.1. *$(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{C}^n$ lies in $\sigma(x_1, \ldots, x_n)$ if and only if the equation*

$$\sum_{j=1}^n y_j(x_j - \lambda_j) = 1$$

has no solution $y_1, \ldots, y_n \in \mathfrak{A}$.

We shall prove

Theorem 8.2. *Fix $x_1, \ldots, x_n \in \mathfrak{A}$. Let Ω be an open set in $\mathbb{C}^n$ with $\sigma(x_1, \ldots, x_n) \subset \Omega$. For each $F \in H(\Omega)$ there exists $y \in \mathfrak{A}$ with*

(2) $$\hat{y}(M) = F(\hat{x}_1(M), \ldots, \hat{x}_n(M)), \qquad \textit{all } M \in \mathcal{M}.$$

Remark. This result is, of course, not a full generalization of Theorem 3.3. We shall see that it is adequate for important applications, however, When $\mathfrak{A}$ is semisimple, we can say more. In that case y is determined uniquely by (2) and we can define

$$F(x_1, \ldots, x_n) = y.$$

Now $H(\Omega)$ is an F-space in the sense of [DSch, Chap. II]. Hence by the closed graph theorem (*loc. cit.*), the map

$$F \to F(x_1, \ldots, x_n)$$

is continuous from $H(\Omega) \to \mathfrak{A}$. Thus

Corollary. *If $\mathfrak{A}$ is semisimple, $F_j \to F$ in $H(\Omega)$ implies that $F_j(x_1, \ldots, x_n) \to F(x_1, \ldots, x_n)$ in $\mathfrak{A}$.*

We shall first prove our theorem under the assumption that

$$\begin{array}{ll} & x_1, \ldots, x_n \text{ generate } \mathfrak{A}\text{; i.e., the smallest closed subalgebra of } \mathfrak{A} \\ (3) & \text{containing} \quad x_1, \ldots, x_n \text{ coincides with } \mathfrak{A}. \end{array}$$

Lemma 8.3. *Assume (3). Then $\sigma(x_1, \ldots, x_n)$ is a polynomially convex subset of $\mathbb{C}^n$.*

PROOF. Fix $z^0 = (z_1^0, \ldots, z_n^0)$ with

$$|Q(z^0)| \le \max_\sigma |Q|, \qquad \text{all polynomials } Q,$$

where $\sigma = \sigma(x_1, \ldots, x_n)$.

$$\begin{aligned} \max_\sigma |Q| &= \max_{\mathcal{M}} |Q(\hat{x}_1, \ldots, \hat{x}_n)| = \max_{\mathcal{M}} |\widehat{Q(x_1, \ldots, x_n)}| \\ &\le \|Q(x_1, \ldots, x_n)\|. \end{aligned}$$

Hence the map $\chi : Q(x_1, \ldots, x_n) \to Q(z^0)$ is a bounded homomorphism from a dense subalgebra of $\mathfrak{A} \to \mathbb{C}$. (Check that χ is unambiguously defined.) Hence χ extends to a homomorphism of $\mathfrak{A} \to \mathbb{C}$, so $\exists M_0 \in \mathcal{M}$ with $\chi(f) = \hat{f}(M_0)$, all $f \in \mathfrak{A}$. In particular,

$$\chi(x_j) = \hat{x}_j(M_0) \text{ or } z_j^0 = \hat{x}_j(M_0), \qquad j = 1, \ldots, n.$$

Thus $z^0 \in \sigma$. Hence σ is polynomially convex. □

EXERCISE 8.1. Let F be holomorphic in a neighborhood of Δ^N with

$$F(\zeta) = \sum_\nu C_\nu \zeta_1^{\nu_1} \cdots \zeta_N^{\nu_N}.$$

Given that $y_1, \ldots, y_N \in \mathfrak{A}$, $\max_{\mathcal{M}} |\hat{y}_j| \le 1$, all j. Then

$$\sum_\nu C_\nu y_1^{\nu_1} \cdots y_N^{\nu_N}$$

converges in $\mathfrak{A}$.

PROOF OF THEOREM 8.2, ASSUMING (3). Without loss of generality, $||x_j|| \leq 1$ for all j By Lemma 8.3, $\sigma = \sigma(x_1, \ldots, x_n)$ is polynomially convex, and $\sigma \subset \Delta^n$. By Lemma 7.4, $\exists$ a p-polyhedron $\prod$ with $\sigma \subset \prod \subset \Omega$, $\prod = P^n(p_1, \ldots, p_r)$. Fix $\phi \in H(\Omega)$. By the Oka extension theorem, $\exists \Phi$ holomorphic in a neighborhood of Δ^{n+r} with

$$\Phi(z_1, \ldots, z_n, p_1(z), \ldots, p_r(z)) = \phi(z), \qquad z \in \prod .$$

Put $y_1 = x_1, \ldots, y_n = x_n, y_{n+1} = p_1(x_1, \ldots, x_n), \ldots, y_{n+r} = p_r(x_1, \ldots, x_n)$. We verify that $\max_{\mathcal{M}} |\hat{y}_j| \leq 1$, $j = 1, 2, \ldots, n + r$. By Exercise 8.1,

$$\sum_\nu C_\nu x_1^{\nu_1} \cdots {}_n^{\nu_n} (p_1(x))^{\nu_{n+1}} (p_r(x))^{\nu_{n+r}}$$

converges in $\mathfrak{A}$ to an element y, where $\sum_\nu C_\nu \zeta^\nu$ is the Taylor expansion of Φ at 0 and $p_j(x)$ denotes $p_j(x_1, \ldots, x_n)$. Then

$$\begin{aligned} \hat{y}(M) &= \Phi(\hat{x}_1(M), \ldots, \hat{x}_n(M), p_1(\hat{x}(M)), \ldots, p_r(\hat{x}(M))) \\ &= \phi(\hat{x}_1(M), \ldots, \hat{x}_n(M)), \qquad \text{all } M \in \mathcal{M}, \end{aligned}$$

since $(\hat{x}_1(M), \ldots, \hat{x}_n(M)) \in \sigma \subset \prod$. We are done.

If we now drop (3), σ is no longer polynomially convex. Richard Arens and Alberto Calderon fortunately found a way to reduce the general case to the finitely generated one.

Let $x_1, \ldots, x_n \in \mathfrak{A}$, let W be an open set in $\mathbb{C}^n$ containing $\sigma(x_1, \ldots, x_n)$, and fix $F \in H(W)$. For every closed subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ containing elements $\zeta_1, \ldots, \zeta_k$ of $\mathfrak{A}$, let $\sigma_{\mathfrak{A}'}(\zeta_1, \ldots, \zeta_k)$ denote the joint spectrum of $\zeta_1, \ldots, \zeta_k$ relative to $\mathfrak{A}'$.

Assertion. $\exists C_1, \ldots, C_m \in \mathfrak{A}$ such that if B is the closed subalgebra of $\mathfrak{A}$ generated by $x_1, \ldots, x_n, C_1, \ldots, C_m$, then

$$\sigma_B(x_1, \ldots, x_n) \subset W. \tag{4}$$

Grant this for now. Let π be the projection $z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) \to (z_1, \ldots, z_n)$ of $\mathbb{C}^{n+m} \to \mathbb{C}^n$. Because of (4), $\sigma_B(x_1, \ldots, x_n, C_1, \ldots, C_m) \subset \pi^{-1}(W)$. Define a function ϕ on $\pi^{-1}(W)$ by

$$\phi(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{n+m}) = F(z_1, \ldots, z_n).$$

Thus ϕ is holomorphic in a neighborhood of $\sigma_B(x_1, \ldots, x_n, C_1, \ldots, C_m)$, and so, by Theorem 8.2 under hypothesis (3) applied to B and the set of generators $x_1, \ldots, C_m$, $\exists y \in B$ with

$$\begin{aligned} \hat{y} &= \phi(\hat{x}_1, \ldots, \hat{x}_n, \hat{C}_1, \ldots \hat{C}_m) \\ &= F(\hat{x}_1, \ldots, \hat{x}_n) \text{ on } \mathcal{M}(B). \end{aligned}$$

If $M \in \mathcal{M}$, then $M \cap B \in \mathcal{M}(B)$ and hence $\hat{y}(M) = F(\hat{x}_1(M), \ldots, \hat{x}_n(M))$. We are done, except for the proof of the assertion.

Let $\mathfrak{A}_0$ denote the closed subalgebra generated by $x_1, \ldots, x_n$ and put $\sigma_0 = \sigma_{\mathfrak{A}_0}(x_1, \ldots, x_n)$. If $\sigma_0 \subset W$, take $B = \mathfrak{A}_0$. If not, consider $\zeta \in \sigma_0 \backslash W$.

Since $\zeta \notin \sigma(x_1, \ldots, x_n)$, $\exists y_1, \ldots, y_n \in \mathfrak{A}$ such that $\sum_{j=1}^n y_j(x_j - \zeta_j) = 1$. Denote by $\mathfrak{A}(\zeta)$ the closed subalgebra generated by $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then $\exists$ neighborhood $\mathcal{N}_\zeta$ of ζ in $\mathbb{C}$ such that if $\alpha \in \mathcal{N}_\zeta$, then $\sum_j y_j(x_j - \alpha_j)$ is invertible in $\mathfrak{A}(\zeta)$. It follows that if $\alpha \in \mathcal{N}_\zeta$, then $\alpha \notin \sigma_{\mathfrak{A}(\zeta)}(x_1, \ldots, x_n)$.

By compactness of $\sigma_0 \backslash W$, we obtain in this way a finite covering of $\sigma_0 \backslash W$ by neighborhoods $\mathcal{N}_\zeta$. We throw together all the corresponding y_j and call them $C_1, \ldots, C_m$, and we let B be the closed subalgebra generated by $x_1, \ldots, x_n, C_1, \ldots, C_m$. Note that $\sigma_B(x_1, \ldots, x_n) \subset \sigma_0$. (Why?) If $\alpha \in \sigma_0 \backslash W$, then α lies in one of our finitely many $\mathcal{N}_\zeta$, and so $\exists u_1, \ldots, u_n \in B$ such that $\sum_j u_j(x_j - \alpha_j)$ is invertible in B. Hence $\alpha \notin \sigma_B(x_1, \ldots, x_n)$. Thus $\sigma_B(x_1, \ldots, x_n) \subset W$, proving the assertion. Thus Theorem 8.2 holds in general.

As a first application we consider this problem. Let $\mathfrak{A}$ be a Banach algebra and $x \in \mathfrak{A}$. When does x have a square root in $\mathfrak{A}$, i.e., when we can find $y \in \mathfrak{A}$ with $y^2 = x$?

An obvious necessary condition is the purely topological one:

$$\exists y \in C(\mathcal{M}) \qquad \text{with } y^2 = \hat{x} \text{ on } \mathcal{M}. \tag{5}$$

Condition (5) alone is not sufficient, as is seen by taking, with $D = \{z \big| |z| \leq 1\}$,

$$\mathfrak{A} = \{f \in A(D) | f'(0) = 0\}.$$

Then $z^2 \in \mathfrak{A}$, $z \notin \mathfrak{A}$, but (5) holds. However, one can prove

Theorem 8.4. *Let $\mathfrak{A}$ be a Banach algebra, $a \in \mathfrak{A}$. and assume that $\exists h \in C(\mathcal{M})$ with $h^2 = \hat{a}$. Assume also that $\hat{a}$ never vanishes on $\mathcal{M}$. Then a has a square root in $\mathfrak{A}$.*

We approach the proof as follows: First find $a_2, \ldots, a_n \in \mathfrak{A}$ such that $\exists F$ holomorphic in a neighborhood of $\sigma(a, a_2, \ldots, a_n)$ in $\mathbb{C}^n$ with $F^2 = z_1$. By Theorem 8.2, $\exists y \in \mathfrak{A}$, with $\hat{y} = F(\hat{a}, \hat{a}_2, \ldots, \hat{a}_n)$ on $\mathcal{M}$. Then $\hat{y}^2 = \hat{a}$ on $\mathcal{M}$. If $\mathfrak{A}$ is semisimple, we are done. In the general case, put $\rho = a - y^2$. Then $\rho \in \operatorname{rad} \mathfrak{A}$. Since $\hat{y}^2 = \hat{a}$, y^2 is invertible and $\rho/y^2 \in \operatorname{rad} \mathfrak{A}$. Then $(y\sqrt{1 + \rho/y^2})^2 = y^2(1 + \rho/y^2) = a$, so $y\sqrt{1 + \rho/y^2}$ solves our problem provided that $\sqrt{1 + \rho/y^2} \in \mathfrak{A}$. It does so by

Exercise 8.2. Let $\mathfrak{A}$ be a Banach algebra and $x \in \operatorname{rad} \mathfrak{A}$. Then $\exists \zeta \in \mathfrak{A}$ with $\zeta^2 = 1 + x$ and $\hat{\zeta} \equiv 1$ on $\mathcal{M}$.

We return to the details.

Lemma 8.5. *Given a as in Theorem 8.4, $\exists a_2, \ldots, a_n \in \mathfrak{A}$ such that if $K = \sigma(a, a_2, \ldots, a_n) \subset \mathbb{C}^n$, then we can find $H \in C(K)$ with $H^2 = z_1$ on K.*

PROOF. In the topological product $\mathcal{M} \times \mathcal{M}$ put

$$S = \{(M, M')|h(M) + h(M') = 0\},$$

where h is as in Theorem 8.4. S is compact and disjoint from the diagonal. (Why?) Let $x = (M_1, M_1') \in S$. Since $M_1' \neq M_1$, $\exists b_x \in \mathfrak{A}$ with $\widehat{b_x}(M_1) - \widehat{b_x}(M_1') \neq 0$. By continuity $\widehat{b_x}(M) - \widehat{b_x}(M') \neq 0$ for all (M, M') in some neighborhood $\mathcal{N}_x$ of x in S. By compactness, $\mathcal{N}_{x_2}, \ldots, \mathcal{N}_{x_n}$ cover S for a suitable choice of $x_2, \ldots, x_n$. Put $a_j = b_{x_j}$, $j = 2, \ldots, n$. Put

$$K = \sigma(a, a_2, \ldots, a_n)$$

and fix $z = (\hat{a}(M), \hat{a}_2(M), \ldots, \hat{a}_n(M)) \in K$.

We define a function H on K by $H(z) = h(M)$. To see that H is well defined, suppose that for $(M, M') \in \mathcal{M} \times \mathcal{M}$,

$$\hat{a}(M) = \hat{a}(M'), \hat{a}_j(M) = \hat{a}_j(M'), \ j = 2, \ldots, n. \tag{6}$$

$(M, M') \notin S$, for this would imply that $(M, M') \in \mathcal{N}_{x_j}$ for some j, denying (6). Hence $h(M) \neq -h(M')$. By (6), $h^2(M) = h^2(M')$. Hence $h(M) = h(M')$, as desired. It is easily verified that H is continuous on K, and that $H^2 = z_1$. □

PROOF OF THEOREM 8.4. It only remains to construct F holomorphic in a neighborhood of K with $F^2 = z_1$.

For each $x \in K$ and $r > 0$, let $B(x, r)$ be the open ball in $\mathbb{C}^n$ centered at x and of radius r. If $x = (\alpha_1, \ldots, \alpha_n) \in K$, $\alpha_1 \neq 0$. Hence $\exists r > 0$ and F_x holomorphic in $B(x, r)$, with $F_x^2 = z_1$ in $B(x, r)$. By compactness of K, a fixed r will work for all x in K. This is not enough, however, to yield an F holomorphic in a neighborhood of K with $F^2 = z_1$. (Why not?) But we can require, in addition, that $F_x = H$ in $B(x, r) \cap K$. Put $\Omega = \bigcup_{x \in K} B(x, r/2)$. For $\zeta \in \Omega$, define $F(\zeta) = F_x(\zeta)$ if $\zeta \in B(x, r/2)$, $x \in K$. To see that this value is independent of $x \in K$, suppose that $\zeta \in B(x, r/2) \cap B(y, r/2)$, $x, y \in K$.

Then $y \in B(x, r) \cap K$. Hence $F_x(y) = H(y)$. Also, $F_y(y) = H(y)$. Hence F_x and F_y are two holomorphic functions in $B(x, r) \cap B(y, r/2)$ with $F_x^2 = F_y^2 = z_1$ there and $F_x = F_y$ at y. So $F_x(\zeta) = F_y(\zeta)$. (Why?) Thus $F \in H(\Omega)$ and $F^2 = z_1$ in Ω. □

Theorem 8.4 holds when the square-root function is replaced by any one of a large class of multivalued analytic functions. (See the Notes at the end of this section.)

As our second application of Theorem 8.2, we take the existence of idempotent elements.

Theorem 8.6 (Šilov Idempotent Theorem). *Let $\mathfrak{A}$ be a Banach algebra and assume that $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are disjoint closed sets. Then $\exists e \in \mathfrak{A}$ with $e^2 = e$ and $\hat{e} = 1$ on $\mathcal{M}_1$ and $\hat{e} = 0$ on $\mathcal{M}_2$.*

Lemma 8.7. *$\exists a_1, \ldots, a_N \in \mathfrak{A}$ such that if $\hat{a}$ is the map of $\mathcal{M} \to C^N : M \to (\hat{a}_1(M), \ldots, \hat{a}_N(M))$, then $\hat{a}(\mathcal{M}_1) \cap \hat{a}(\mathcal{M}_2) = \emptyset$.*

The proof is like that of Lemma 8.5 and is left to the reader.

Proof of Theorem 8.6. By the last Lemma, $\exists a_1, \ldots, a_N \in \mathfrak{A}$, so that $\hat{a}(\mathcal{M}_1)$ and $\hat{a}(\mathcal{M}_2)$ are disjoint compact subsets of $\mathbb{C}^N$. Choose disjoint open sets W_1 and W_2 in $\mathbb{C}^N$ with $\hat{a}(\mathcal{M}_j) \subset W_j$, $j = 1, 2$. Put $W = W_1 \cup W_2$ and define F in W by $F = 1$ on W_1 and $F = 0$ on W_2. Then $F \in H(W)$. By Theorem 8.2, $\exists t \in \mathfrak{A}$ with $\hat{y} = F(\hat{a}_1, \ldots, \hat{a}_n)$ on $\mathcal{M}$. Then $\hat{y} = 1$ on $\mathcal{M}_1$, $\hat{y} = 0$ on $\mathcal{M}_2$. We seek $u \in \operatorname{rad} \mathfrak{A}$ so that $(y+u)^2 = y + u$. Then $e = y + u$ will be the desired element.

The condition on $u \Leftrightarrow$

$$u^2 + (2y - 1)u + \rho = 0, \tag{7}$$

where $\rho = y^2 - y \in \operatorname{rad} \mathfrak{A}$.

The formula for solving a quadratic equation suggests that we set

$$u = -\frac{2y-1}{2} + \frac{2y-1}{2}\zeta,$$

where ζ is the element of $\mathfrak{A}$, provided by Exercise 8.2, satisfying

$$\zeta^2 = 1 - \frac{4\rho}{(2y-1)^2} \qquad \text{and} \qquad \hat{\zeta} \equiv 1.$$

We can then check that u has the required properties, and the proof is complete.

Corollary 1. *If $\mathcal{M}$ is disconnected, $\mathfrak{A}$ contains a nontrivial idempotent.*

Corollary 2. *Let $\mathfrak{A}$ be a uniform algebra on a compact space X. Assume that $\mathcal{M}$ is totally disconnected. Then $\mathfrak{A} = C(X)$.*

Note. The hypothesis is on $\mathcal{M}$, not on X, but it follows that if $\mathcal{M}$ is totally disconnected, then $\mathcal{M} = X$.

Proof of Corollary 2. If $x_1, x_2 \in X$, $x_1 \neq x_2$, choose an open and closed set $\mathcal{M}_1$ in $\mathcal{M}$ with $x_1 \in \mathcal{M}_1$, $x_2 \notin \mathcal{M}_1$. Put $\mathcal{M}_2 = \mathcal{M} \backslash \mathcal{M}_1$. By Theorem 8.6, $\exists e \in \mathfrak{A}$, $\hat{e} = 1$ on $\mathcal{M}_1$ and $\hat{e} = 0$ on $\mathcal{M}_2$. Thus e is a real-valued function in $\mathfrak{A}$ which separates x_1 and x_2. By the Stone-Weierstrass theorem, we conclude that $\mathfrak{A} = C(X)$.

Corollary 3. *Let X be a compact subset of $\mathbb{C}^n$. Assume that X is polynomially convex and totally disconnected. Then $P(X) = C(X)$.*

Proof. The result follows from Corollary 2, together with the fact that $\mathcal{M}(P(X)) = X$.

NOTES

Theorem 8.2 was proved for finitely generated algebras by G. E. Šilov, On the decomposition of a commutative normed ring into a direct sum of ideals, *A.M.S. Transl.* **1** (1955). The proof given here is due to L. Waelbroeck, Le Calcul symbolique dans les algèbres commutatives, *J. Math. Pure Appl.* **33** (1954), 147-186. Theorem 8.2 for the general case was proved by R. Arens and A. Calderon, Analytic functions of several Banach algebra elements, *Ann. Math.* **62** (1955), 204-216. Theorem 8.4 is a special case of a more general result given by Arens and Calderon, *loc. cit.* Theorem 8.6 and its corollaries are due to Šilov, *loc. cit.*

Our proof of Theorem 8.4 has followed Hörmander's book [Hö, Chap. 3].

For a stronger version of Theorem 8.2 see Waelbroeck, *loc. cit.*, or N. Bourbaki, *Théories spectrales*, Hermann, Paris, 1967, Chap. 1, Sec. 4.

9

The Šilov Boundary

Let X be a compact space and $\mathcal{F}$ an algebra of continuous complex-valued functions on X which separates the points of X.

Definition 9.1. A *boundary for* $\mathcal{F}$ is a closed subset E of X such that

$$|f(x)| \leq \max_E |f|, \qquad \text{all } f \in \mathcal{F}, x \in X.$$

Thus, for example, if D is the closed unit disk in $\mathbb{C}$ and $\mathcal{P}$ the algebra of all polynomials in z, restricted to D, then every closed subset of D containing $\{z \mid |z| = 1\}$ is a boundary for $\mathcal{P}$.

Theorem 9.1. *Let X and $\mathcal{F}$ be as above. Let S denote the intersection of all boundaries for $\mathcal{F}$. Then S is a boundary for $\mathcal{F}$.*

Note.

(a) It is not clear, a priori, that S is nonempty.
(b) S is evidently closed.
(c) It follows from the theorem that S is the smallest boundary, i.e., that S is a boundary contained in every other boundary.

Lemma 9.2. *Fix $x \in X \backslash S$. $\exists$ a neighborhood U of x with the following property: If β is a boundary, then $\beta \backslash U$ is also a boundary.*

Proof. $x \notin S$ and so $\exists$ boundary S_0 with $x \notin S_0$. For each $y \in S_0$, choose $f_y \in \mathcal{F}$ with $f_y(x) = 0$, $f_y(y) = 2$.

$\mathcal{N}_y = \{|f_y| > 1\}$ is a neighborhood of y. Then $\exists y_1, \dots, y_k$ so that $\mathcal{N}_{y_1} \cup \cdots \cup \mathcal{N}_{y_k} \supset S_0$. Write f_j for f_{y_j}. Put

$$U = \{|f_1| < 1, \dots, |f_k| < 1\}.$$

Then U is a neighborhood of x and $U \cap S_0 = \emptyset$

Fix a boundary β and suppose that $\beta \backslash U$ fails to be a boundary. Then $\exists f \in \mathcal{F}$ $\max_x |f| = 1$, with $\max_{\beta \backslash U} |f| < 1$.

Assertion. $\exists n$ so that $\max_X |f^n f_i| < 1, i = 1, \ldots, k$.

Grant this for now. Since S_0 is a boundary, we can pick $\bar{x} \in S_0$ with $|f(\bar{x})| = 1$. By the assertion, $|f_i(\bar{x})| < 1, i = 1, \ldots, k$.

Hence $\bar{x} \in U$, denying $U \cap S_0 = \emptyset$. Thus $\beta \backslash U$ is a boundary, and we are done.

To prove the assertion, fix M with $\max_X |f_i| < M, i = 1, \ldots, k$. Chose n so that $(\max_{\beta\backslash U} |f|)^n \cdot M < 1$. Then $|f^n f_i| < 1$ at each point $\beta\backslash U$ for every i. On U, $|f^n f_i| < 1$ by choice of U. Hence the assertion.

PROOF OF THEOREM 9.1. Let W be an open set containing S. For each $x \in X\backslash W$ construct a neighborhood U_x by Lemma 9.2. $X\backslash W$ is compact, so we can find finitely many such U_x, say $U_1, \ldots, U_r$, whose union covers $X\backslash W$.

X is a boundary. By choice of U_1, $X\backslash U_1$ is a boundary. Hence $(X/U_1)\backslash U_2$ is a boundary, and at last $X^* = X\backslash(U_1 \cup U_2 \cup \cdots \cup U_r)$ is a boundary. But $X^* \subseteq W$. Hence if $f \in \mathcal{F}$, $\max_X |f| \leq \sup_W |f|$. Since W was an arbitrary neighborhood of S, it follows that S is a boundary. (Why?) □

Note. What properties of $\mathcal{F}$ were used in the proof?

Let $\mathfrak{A}$ be a Banach algebra. Then $\hat{\mathfrak{A}}$ is an algebra of continuous functions on $\mathcal{M}$, separating points. By Theorem 9.1 $\exists a$ (unique) boundary S for $\hat{\mathfrak{A}}$ which is contained in every boundary.

Definition 9.2. S is called the *Šilov boundary* of $\mathfrak{A}$ and is denoted $\check{S}(\mathfrak{A})$.

EXERCISE 9.1. Let Ω be a bounded plane region whose boundary consists of finitely many simple closed curves. Then $\check{S}(A(\Omega))$ = topological boundary $\partial\Omega$ of Ω.

EXERCISE 9.2. Let Y denote the solid cylinder = $\{(z, t) \in \mathbb{C} \times \mathbb{R} \big| |z| \leq 1, 0 \leq t \leq 1\}$. Let $\mathfrak{A}(Y) = \{f \in C(Y) |$ for each t, $f(z, t)$ is analytic in $|z| < 1\}$. Then $\check{S}(\mathfrak{A}(Y)) = \{(z, t) \big| |z| = 1, 0 \leq t \leq 1\}$.

EXERCISE 9.3. Let Y be as in Exercise 9.2 and put $\mathcal{L}(Y) = \{f \in C(Y) | f(z, 1)$ is analytic in $|z| < 1\}$. Then $\check{S}(\mathcal{L}(Y)) = Y$.

EXERCISE 9.4. Let $\Delta^2 = \{(z, w) \in \mathbb{C}^2 \big| |z| \leq 1, |w| \leq 1\}$ and $\mathfrak{A}(\Delta^2) = \{f \in C(\Delta^2) | f \in H(\Omega)$, where Ω = interior of $\Delta^2\}$. Show that $\check{S}(\mathfrak{A}(\Delta^2)) = T = \{(z, w) | |z| = |w| = 1\}$. Note that here the Šilov boundary is a two-dimensional subset of the three-dimensional topological boundary of Δ^2.

EXERCISE 9.5. Let $B^n = \{z \in \mathbb{C}^n | \sum_{i=1}^n |z_i|^2 \leq 1\}$ and $\mathfrak{A}(B^n) = \{f \in C(B^n) | f \in H(\Omega), \Omega$ = interior of $B^n\}$. Show that $\check{S}(\mathfrak{A}(B^n))$ = topological boundary of B^n.

Note that in all these examples, as well as in many others arising naturally, the complement $\mathcal{M}\backslash\check{S}(\mathfrak{A})$ of the Šilov boundary in the maximal ideal space is the union of one or many complex-analytic varieties, and the elements of $\hat{\mathfrak{A}}$ are analytic when restricted to these varieties.

We shall study this phenomenon of "analytic structure" in $\mathcal{M}\backslash\check{S}(\mathfrak{A})$ in several later sections.

We now proceed to consider one respect in which elements of $\hat{\mathfrak{A}}$ *act like* analytic functions on $\mathcal{M}\backslash\check{S}(\mathfrak{A})$.

Let Ω be a bounded domain in $\mathbb{C}$. We have

(1) $$\text{For } F \in \mathfrak{A}(\Omega),\ x \in \Omega,\ |F(x)| \leq \max_{\partial\Omega} |F|.$$

The analogous inequality for an arbitrary Banach algebra $\mathfrak{A}$ is true by definition: For $f \in \mathfrak{A}$, $x \in \mathcal{M}$,

$$|\hat{f}(x)| \leq \max_{\check{S}(\mathfrak{A})} |\hat{f}|.$$

However, we also have a *local* statement for $\mathfrak{A}(\Omega)$. Fix $x \in \Omega$ and let U be a neighborhood of x in Ω. Then

(2) $$\text{For } F \in \mathfrak{A}(\Omega),\ |F(x)| \leq \max_{\partial U} |F|.$$

The analogue of (2) for arbitrary Banach algebras is by no means evident. It is, however, true.

Theorem 9.3 (Local Maximum Modulus Principle). *Let $\mathfrak{A}$ be a Banach algebra and fix $x \in \mathcal{M}\backslash\check{S}(\mathfrak{A})$. Let U be a neighborhood of x with $U \subset \mathcal{M}\backslash\check{S}(\mathfrak{A})$. Then for all $f \in \mathfrak{A}$,*

(3) $$|\hat{f}(x)| \leq \max_{\partial U} |\hat{f}|.$$

Lemma 9.4. *Let X be a compact, polynomially convex set in $\mathbb{C}^n$ and U_1 and U_2 be open sets in $\mathbb{C}^n$ with $X \subset U_1 \cup U_2$. If $h \in H(U_1 \cap U_2)$, then $\exists$ a neighborhood W of X and $h_j \in H(W \cap U_j)$, $j = 1, 2$, so that*

$$h_1 - h_2 = h \text{ in } W \cap U_1 \cap U_2.$$

Proof. Write $X = X_1 \cup X_2$, where X_j is compact and $X_j \subset U_j$, $j = 1, 2$. Choose $f_1 \in C_0^\infty(U_1)$ with $0 \leq f_1 \leq 1$ and $f_1 = 1$ on X_1. Similarly, choose $f_2 \in C_0^\infty(U_2)$. Then $f_1 + f_2 \geq 1$ on X, and so $f_1 + f_2 > 0$ in a neighborhood V of X. In V define

$$\eta_1 = \frac{f_1}{f_1 + f_2}, \qquad \eta_2 = \frac{f_2}{f_1 + f_2}.$$

Then $\eta_1, \eta_2 \in C^\infty(V)$, $\eta_1 + \eta_2 = 1$ in V, and $\operatorname{supp} \eta_j \subset U_j$, $j = 1, 2$. With no loss of generality, $U_j = U_j \cap V$. Define functions H_j in $C^\infty(U_j)$, $j = 1, 2$ by

$$H_1 = \eta_2 h \text{ in } U_1 \cap U_2, \qquad H_1 = 0 \text{ in } U_1\backslash U_2.$$
$$H_2 = -\eta_1 h \text{ in } U_1 \cap U_2, \qquad H_2 = 0 \text{ in } U_2\backslash U_1.$$

Then

$$H_1 - H_2 = (\eta_1 + \eta_2)h = h \text{ in } U_1 \cap U_2.$$

Hence $\bar{\partial} H_1 = \bar{\partial} H_2$ in $U_1 \cap U_2$. Let f be the (0, 1)-form in $U_1 \cup U_2$ defined by $f = \bar{\partial} H_1$ in U_1, $f = \bar{\partial} H_2$ in U_2. Then f is $\bar{\partial}$-closed in $U_1 \cup U_2$. We can choose a p-polyhedron $\prod$ with $X \subset \prod \subset U_1 \cup U_2$. By Theorem 7.6, then, $\exists$ a neighborhood W of $\prod$ and $F \in C^\infty(W)$ with $\bar{\partial} F = f$ in W.

Put $h_j = H_j - F$ in $U_j \cap W$, $j = 1, 2$. Then $h_1 - h_2 = h$ in $U_1 \cap U_2 \cap W$, and $\bar{\partial} h_j = f - f = 0$ in $U_j \cap W$; so $h_j \in H(U_j \cap W)$, $j = 1, 2$. □

Lemma 9.5. *Let K be a compact set in $\mathbb{C}^N$ and U_1 and U_2 open sets with*

$$U_1 \cup U_2 \supset K, \tag{4}$$

$$U_1 \cap U_2 \subset \{\operatorname{Re} z_1 < 0\} \quad \textit{and} \quad \exists\, h_1 \in H(U_1),\, h_2 \in H(U_2) \tag{5}$$

with

$$h_1 - h_2 = \frac{\log z_1}{z_1} \textit{ in } U_1 \cap U_2 \quad \textit{and} \quad K \cap U_2 \subset \{\operatorname{Re} z_1 \leq 0\}. \tag{6}$$

Then $\exists F$ holomorphic in a neighborhood of K with $F = 1$ on $K \cap \{z_1 = 0\} \cap U_2$ and $|F| < 1$ elsewhere on K.

Proof. By (5) we have in $U_1 \cap U_2$,

$$z_1 h_1 - z_1 h_2 = \log z_1 \quad \text{so} \quad e^{z_1 h_1} = z_1 e^{z_1 h_2}.$$

It follows that if we define

$$f = \begin{cases} e^{z_1 h_1} & \text{in } U_1, \\ z_1 e^{z_1 h_2} & \text{in } U_2, \end{cases}$$

then $f \in H(U_1 \cup U_2)$. Also

$$f \text{ never vanishes on } K \backslash (\{z_1 = 0\} \cap U_2). \tag{7}$$

Assertion. $\exists \varepsilon > 0$ such that if $z \in K \backslash (\{z_1 = 0\} \cap U_2)$, then $f(z)$ lies outside the disk $\{|w - \varepsilon| \leq \varepsilon\}$.

Assume first that $z \in U_2$. Then

$$z_1 = e^{-z_1 h_2} f, \text{ so } z_1 h_2 = e^{-z_1 h_2} \cdot h_2 f,$$

or $z_1 h_2 = C \cdot f$, with $C \in H(U_2)$. Hence $z_1 = f e^{-Cf} = f + kf^2$, with $k \in H(U_2)$. By shrinking U_2 we may obtain $|k| \leq M$ on U_2, M a constant. Since $\operatorname{Re} z_1 \leq 0$ by (6), we have, at z,

$$\begin{aligned} 0 &\geq \operatorname{Re} f + \operatorname{Re}(kf^2) \geq \operatorname{Re} f - |f|^2 |k| \\ &\geq \operatorname{Re} f - M|f|^2. \end{aligned}$$

Put $f(z) = w = u + iv$. Then

$$u - M(u^2 + v^2) \le 0,$$

and so

$$\left(u - \frac{1}{2M}\right)^2 + v^2 \ge \frac{1}{4M^2}.$$

Thus $f(z)$ lies outside the disk:

$$|w - \frac{1}{2M}| < \frac{1}{2M}.$$

On the other hand, $K\backslash U_2$ is compact and $f \neq 0$ there. Hence for some $r > 0$, $|f(z)| \ge r$ if $z \in K\backslash U_2$. The assertion now follows.

Let D_ε be the disk $\{|w - \varepsilon| \le \varepsilon\}$ just obtained and put

$$F = -\frac{\varepsilon}{f - \varepsilon}.$$

By choice of D_ε, F is holomorphic in some neighborhood of K. Also on $\{z_1 = 0\} \cap U_2$, $F = 1$ since $f = 0$, and everywhere else on K, $|F| < 1$ since $|f - \varepsilon| > \varepsilon$. □

Lemma 9.6. *Let $\mathfrak{A}$ be a Banach algebra, T a closed subset of $\mathcal{M}$ and U an open neighborhood of T. Suppose that $\exists \phi \in \mathfrak{A}$ with $\hat{\phi} = 1$ on T, $|\hat{\phi}| < 1$ on $U\backslash T$. Then $\exists \Phi \in \mathfrak{A}$ with $\hat{\Phi} = 1$ on T, $|\hat{\Phi}| < 1$ on $\mathcal{M}\backslash T$.*

PROOF. T and $\mathcal{M}\backslash U$ are disjoint closed subsets of $\mathcal{M}$. Hence $\exists g_2, \ldots, g_n \in \mathfrak{A}$ such that if $\hat{g} : \mathcal{M} \to \mathbb{C}^{n-1}$ is the map $m \to (\hat{g}_2(m), \ldots, \hat{g}_n(m))$, then $\hat{g}(T) \cap \hat{g}(\mathcal{M}\backslash U) = \emptyset$. (Why?)

Put $g_1 = \phi - 1$. Then $\hat{g}_1 = 0$ on T and $\operatorname{Re} \hat{g}_1 < 0$ on $U\backslash T$. Let now $G : \mathcal{M} \to \mathbb{C}^n$ be the map sending $m \to (\hat{g}_1(m), \hat{g}_2(m), \ldots, \hat{g}_n(m))$. Then $G(\mathcal{M}) = \sigma(g_1, \ldots, g_n)$. We have

(8) $$G(T) \text{ is a compact subset of } \{z_1 = 0\},$$

(9) $$G(T) \text{ is disjoint from } G(\mathcal{M}\backslash U),$$

(10) $$G(U\backslash T) \subset \{\operatorname{Re} z_1 < 0\}.$$

Choose a neighborhood Δ of $G(T)$ in $\mathbb{C}^n$ with $\bar{\Delta} \cap G(\mathcal{M}\backslash U) = \emptyset$. It is easily seen that $\exists$ an open set D_0 in $\mathbb{C}^n$ such that

(11) $$D_0 \cup \Delta \supset G(\mathcal{M}) \quad \text{and} \quad D_0 \cap \Delta \subset \{\operatorname{Re} z_1 < 0\}.$$

By a construction used in the proof of Theorem 8.2, $\exists C_1, \ldots, C_m \in \mathfrak{A}$ such that if B is the closed subalgebra generated by $g_1, \ldots, g_n, C_1, \ldots, C_m$, then $\sigma_B(g_1, \ldots, g_n) \subset D_0 \cup \Delta$.

Put $\sigma = \sigma(g_1, \ldots, g_n, C_1, \ldots, C_m) \subset \mathbb{C}^{n+m}$, and let $\hat{\sigma}$ be the polynomially convex hull of σ in $\mathbb{C}^{n+m}$. Let π be the natural projection of $\mathbb{C}^{n+m}$ on $\mathbb{C}^n$.

Since $\sigma \subset \sigma_B(g_1, \dots, g_n, C_1, \dots, C_m)$, and since the latter set is polynomially convex because $g_1, \dots, C_m$ generate B, $\hat{\sigma} \subset \sigma_B(g_1, \dots, g_n, C_1, \dots, C_m)$, and so

$$\pi(\hat{\sigma}) \subset \pi(\sigma_B(g_1, \dots, C_m)) = \sigma_B(g_1, \dots, g_n).$$

Thus $\pi(\hat{\sigma}) \subset D_0 \cup \Delta$, and so

$$\hat{\sigma} \subset \pi^{-1}(D_0) \cup \pi^{-1}(\Delta). \tag{12}$$

Because of (11) we have

$$\pi^{-1}(D_0) \cap \pi^{-1}(\Delta) \subset \{\operatorname{Re} z_1 < 0\}. \tag{13}$$

Now $\hat{\sigma}$ is polynomially convex and $(\log z_1)/z_1$ is holomorphic in $\pi^{-1}(D_0) \cap \pi^{-1}(\Delta)$. Lemma 9.4 then yields a neighborhood W of $\hat{\sigma}$, and $h_1 \in H(\pi^{-1}(D_0) \cap W)$, $h_2 \in H(\pi^{-1}(\Delta) \cap W)$ such that

$$h_1 - h_2 = \frac{\log z_1}{z_1} \qquad \text{in } \pi^{-1}(D_0) \cap \pi^{-1}(\Delta) \cap W.$$

We now apply Lemma 9.5 with $\sigma = K$, $U_1 = \pi^{-1}(D_0) \cap W$, and $U_2 = \pi^{-1}(\Delta) \cap W$. Since $\sigma \subseteq \hat{\sigma}$, hypotheses (4) and (5) hold. By choice of Δ and (10), $G(\mathcal{M}) \cap \Delta \subset \{\operatorname{Re} z_1 \leq 0\}$, whence $\sigma \cap \pi^{-1}(\Delta) \subset \{\operatorname{Re} z_1 \leq 0\}$. So hypothesis (6) also holds. We conclude the existence of F holomorphic in a neighborhood of σ with $F = 1$ on $\{z_1 = 0\} \cap \pi^{-1}(\Delta) \cap (\Delta) \cap \sigma$ and $|F| < 1$ elsewhere on σ.

By Theorem 8.2, $\exists \Phi \in \mathfrak{A}$ with

$$\hat{\Phi}(M) = F(\hat{g}_1(M), \dots, \hat{g}_n(M), \hat{C}_1(M), \dots, \hat{C}_m(M))$$

for all $M \in \mathcal{M}$. For $M \in T$, the corresponding point of σ is in $\{z_1 = 0\} \cap \pi^{-1}(\Delta)$, so $\Phi(M) = 1$. For $M \in \mathcal{M} \backslash T$, the corresponding point of σ is not in $\{z_1 = 0\} \cap \pi^{-1}(\Delta)$, so $|\hat{\Phi}(M)| < 1$. □

PROOF OF THEOREM 9.3. Suppose that (3) is false. Chose $x_0 \in \bar{U}$ with $|\hat{f}(x_0)| = \max_{\bar{U}} |\hat{f}|$. Then

$$|\hat{f}(x_0)| > \max_{\partial U} |\hat{f}|. \tag{14}$$

Without loss of generality, $\hat{f}(x_0) = 1$. Let $T = \{y \in \bar{U} | \hat{f}(y) = 1\}$. Then T is compact and $\subset U$. Put $\phi = \frac{1}{2}(1 + f)$. Then $\phi \in \mathfrak{A}$, $\hat{\phi} = 1$ on T, $|\hat{\phi}| < 1$ on $U \backslash T$.

Lemma 9.6 now supplies $\Phi \in \mathfrak{A}$, with $\hat{\Phi} = 1$ on T, $|\hat{\Phi}| < 1$ on $\mathcal{M} \backslash T$. Since $U \subset \mathcal{M} \backslash \check{S}(\mathfrak{A})$, we get that $|\hat{\Phi}| < 1$ on $\check{S}(\mathfrak{A})$. This is impossible, and so (3) holds. □

Note. Some, but not all, of the following exercises depend on Theorem 9.3.

EXERCISE 9.6. Let $\mathfrak{A}$ be a Banach algebra and assume that $\check{S}(\mathfrak{A}) \neq \mathcal{M}$. Show that the restriction of $\hat{\mathfrak{A}}$ to $\check{S}(\mathfrak{A})$ is not uniformly dense in $C(\check{S}(\mathfrak{A}))$.

EXERCISE 9.7. Let $\mathfrak{A}$ be a Banach algebra and assume that $\check{S}(\mathfrak{A}) \neq \mathcal{M}$. Show that $\check{S}(\mathfrak{A})$ is uncountable.

EXERCISE 9.8. Let $\mathfrak{A}$ be a Banach algebra and fix $p \in \check{S}(\mathfrak{A})$. Assume that p is an isolated point of $\check{S}(\mathfrak{A})$, viewed in the topology induced on $\check{S}(\mathfrak{A})$ by $\mathcal{M}$. Show that p is then an isolated point of $\mathcal{M}$.

Theorem 9.7. *Let $\mathfrak{A}$ be a uniform algebra on a space X. Let $U_1, U_2, \ldots, U_s$ be an open covering of $\mathcal{M}$. Denote by $\mathcal{L}$ the set of all f in $C(\mathcal{M})$ such that for $j = 1, \ldots, s$, $f|_{U_j}$ lies in the uniform closure of $\hat{\mathfrak{A}}|_{U_j}$. Then $\mathcal{L}$ is a closed subalgebra of $C(\mathcal{M})$ and $\check{S}(\mathcal{L}) \subseteq X$.*

PROOF. The proof is a corollary of Theorem 9.3. We leave it to the reader as *Exercise 9.9.

EXERCISE 9.10. Is Theorem 9.3 still true if we omit the assumption $U \subset \mathcal{M} \setminus \check{S}(\mathfrak{A})$?

NOTES

Theorem 9.1 is due to G. E. Šilov, On the extension of maximal ideals, *Dokl. Acad. Sci. URSS* (N.S.) (1940), 83-84. The proof given here, which involves no transfinite induction or equivalent argument, is due to Hörmander [Hö2, Theorem 3.1.18]. Theorem 9.3 is due to H. Rossi, The local maximum modulus principle, *Ann. Math.* **72**, No. 1 (1960), 1-11. The proof given here is in the book by Gunning and Rossi [GR, pp. 62-63].

10

Maximality and Radó's Theorem

Let X be a compact space and $\mathfrak{A}$ a uniform algebra on X. Denote by $|| \;\; ||$ the uniform norm on $C(X)$. Note that if $x, y \in \mathfrak{A}$, then $x + \bar{y} \in C(X)$, so that $||x + \bar{y}||$ is defined.

Lemma 10.1 (Paul Cohen). *Let $a, b \in \mathfrak{A}$. Assume that*

$$||1 + a + \bar{b}|| < 1.$$

Then $a + b$ is invertible in $\mathfrak{A}$.

Note. When $b = 0$, this of course holds in an arbitrary Banach algebra.

PROOF. Put $f = a + b$. We have

$$||1 + a + \bar{b}|| < 1, \qquad \text{hence } ||1 + \bar{a} + b|| < 1,$$

whence

$$||1 + a + b + 1 + \bar{a} + b|| < 2 \text{ or } k = ||1 + \operatorname{Re} f|| < 1.$$

For all $x \in X$, then

$$|1 + \operatorname{Re} f(x)| \le k.$$

This means that $f(x)$ lies in the left-half plane for all x, which suggests that for small $\varepsilon > 0$,

$$1 + \varepsilon f(x)$$

lies in the unit disk for all x. Indeed,

$$\begin{aligned} |1 + \varepsilon f(x)|^2 &= 1 + \varepsilon^2 |f(x)|^2 + 2\varepsilon \operatorname{Re} f(x) \\ &\le 1 + c\varepsilon^2 + 2d\varepsilon, \end{aligned}$$

where $c = ||f||^2$ and $d = -1 + k < 0$. Hence for small $\varepsilon > 0$, $|1 + \varepsilon f(x)| < 1$ for all x, or $||1 + \varepsilon f|| < 1$, as we had guessed.

It follows that εf is invertible in $\mathfrak{A}$ for some ε and so f is invertible. $\square$

We shall now apply this lemma to a particular algebra. Let D = closed unit disk in the z-plane and Γ the unit circle. Let $A(D)$ be the space of all functions analytic in $\mathring{D}$ and continuous in D. Put

$$\mathfrak{A}_0 = (A(D)|_\Gamma$$

and give $\mathfrak{A}_0$ the uniform norm on Γ. $\mathfrak{A}_0$ is then isomorphic and isometric to $A(D)$ and is a uniform algebra on Γ. The elements of $\mathfrak{A}_0$ are precisely those functions in $C(\Gamma)$ that admit an analytic extension to $|z| < 1$.

$\mathfrak{A}_0$ is approximately one half of $C(\Gamma)$. For the functions

$$e^{in\theta}, \ n = 0, \pm 1, \pm 2, \ldots$$

span a dense subspace of $C(\Gamma)$, while $\mathfrak{A}_0$ contains exactly those $e^{in\theta}$ with $n \geq 0$.

EXERCISE 10.1. Put $g = \sum_{-p}^{p} c_\nu e^{i\nu\theta}$, where the c_ν are complex constants. Compute the closed algebra generated by $\mathfrak{A}_0$ and g, i.e., the closure in $C(\Gamma)$ of all sums

$$\sum_{\nu=0}^{N} a_\nu g^\nu, \qquad a_\nu \in \mathfrak{A}_0.$$

Theorem 10.2 (Maximality Of $\mathfrak{A}_0$). *Let B be a uniform algebra on Γ with*

$$\mathfrak{A}_0 \subseteq B \subseteq C(\Gamma).$$

Then either $\mathfrak{A}_0 = B$ or $B = C(\Gamma)$.

We shall deduce this result by means of Lemma 10.1 as follows. Assuming $B \neq \mathfrak{A}_0$, we construct elements $u, v \in B$ with

$$||1 + z \cdot u + \bar{z}\bar{v}|| < 1, \tag{1}$$

where $z = e^{i\theta}$. Then we conclude that $zu + zv$ is invertible in B, when z is invertible in B. Hence $B \supset e^{in\theta}$, $n = 0, \pm 1, \pm 2, \ldots$, so $B = C(\Gamma)$, as required. To construct u and v we argue as follows: For each $h \in C(\Gamma)$, put

$$h_k = \frac{1}{2\pi} \int_0^{2\pi} h(e^{i\theta}) e^{-ik\theta} d\theta, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

EXERCISE 10.2. Let $h \in C(\Gamma)$. Prove that $h \in \mathfrak{A}_0$ if and only if $h_k = 0$, for all $k < 0$.

Suppose now that $B \neq \mathfrak{A}_0$. Hence $g \in B$ with $g_k \neq 0$, for some $k < 0$. Without loss of generality we may suppose that $g_{-1} = 1$. (Why?)

Choose a trigonometric polynomial T with

$$||g - T|| < 1. \tag{2}$$

We can assume $T_{-1} = 1$, or

$$T = \sum_{-N}^{-2} T_\nu z^\nu + z^{-1} + \sum_{0}^{N} t_\nu z^\nu.$$

Hence

$$\begin{aligned} zT &= \sum_{-N}^{-2} T_\nu z^{\nu+1} + 1 + z \sum_{0}^{N} T_\nu z^\nu \\ &= \bar{z} \cdot \bar{P} + 1 + zQ, \end{aligned}$$

where P and Q are polynomials in z. Equation (2) gives

$$||zg - zT|| < 1 \qquad \text{or} \qquad ||z(Q - g) + \bar{z}\bar{P} + 1|| < 1.$$

Also $Q - g \in B$, $P \in B$, so we have (1), and we are done. □

Theorem 10.3 (Rudin). *Let $\mathcal{L}$ be an algebra of continuous functions on D such that*
(a) *The function z is in $\mathcal{L}$.*
(b) *$\mathcal{L}$ satisfies a maximum principle relative to Γ:*

$$|G(x)| \leq \max_{\Gamma} |G|, \qquad \textit{all } x \in D, G \in \mathcal{L}.$$

Then $\mathcal{L} \subseteq A(D)$.

Proof. The uniform closure of $\mathcal{L}$ on D, written $\mathfrak{A}$, still satisfies (a) and (b).

Put $B = \mathfrak{A}|_\Gamma$. Because of (b), B is closed under uniform convergence on Γ and by (a), $\mathfrak{A}_0 \subseteq B$. So Theorem 10.2 applies to yield $B = \mathfrak{A}_0$ or $B = C(\Gamma)$.

Consider the map $g \to G(0)$ for $g \in B$, where G is the function in $\mathfrak{A}$ with $G = g$ on Γ. By (b), G is unique. The map is a homomorphism of $B \to \mathbb{C}$ and is not evaluation at a point of Γ. (Why?) Hence $B \neq C(\Gamma)$, and so $B = \mathfrak{A}_0$.

Fix $F \in \mathfrak{A}$. $F|_\Gamma \in \mathfrak{A}_0$, so $\exists F^* \in A(D)$ with $F = F^*$ on Γ. $F - F^*$ then $\in \mathfrak{A}$ and by (b) vanishes identically on D. So $F \in A(D)$ and thus $\mathfrak{A} = A(D)$, whence the assertion. □

Now let X be any compact space, $\mathcal{L}$ an algebra of continuous functions on X, and X_0 a boundary for $\mathcal{L}$ in the sense of Definition 9.1; i.e., X_0 is a closed subset of X with

(3) $$|g(x)| \leq \max_{X_0} |g|, \qquad \text{all } g \in \mathcal{L}, x \in X.$$

Lemma 10.4 (Glicksberg). *Let E be a subset of X_0 and let $f \in \mathcal{L}$ and $f = 0$ on E. Then for each $x \in X$ either*
(a) *$f(x) = 0$, or*
(b) *$|g(x)| \leq \sup_{X_0 \setminus E} |g|$, all $g \in \mathcal{L}$.*

Proof. Fix $g \in \mathcal{L}$. Then $f \cdot g \in \mathcal{L}$. Fix $x \in X$ with $f(x) \neq 0$. We have

$$\begin{aligned} |(fg)(x)| &\leq \max_{X_0} |fg| = \sup_{X_0 \setminus E} |fg| \\ &\leq \sup_{X_0 \setminus E} |f| \cdot \sup_{X_0 \setminus E} |g|. \end{aligned}$$

Hence

$$|g(x)| \leq K \sup_{X_0 \setminus E} |g|,$$

where $K = |f(x)|^{-1} \cdot \sup_{X_0 \setminus E} |f|$. Applying this to g^n, $n = 1, 2, \ldots$ gives

$$|g(x)|^n = |g^n(x)| \leq K \sup_{X_0 \setminus E} |g^n| = K(\sup_{X_0 \setminus E} |g|)^n.$$

Taking nth roots and letting $n \to \infty$ gives (b). □

Consider now the following classical result: Let Ω be a bounded plane region and z_0 a nonisolated boundary point of Ω. Let U be a neighborhood of z_0 in $\mathbb{C}$.

Theorem 10.5. *Let $f \in A(\Omega)$ and assume that $f = 0$ on $\partial\Omega \cap U$. Then $f \equiv 0$ in Ω.*

If we assume that

(4) $$\exists \text{a sequence } \{z_n\} \text{in } \mathbb{C} \setminus \bar{\Omega} \text{with } z_n \to z_0,$$

then Lemma 10.4 gives a direct proof, as follows.

Put $X = \bar{\Omega}$, $\mathcal{L} = A(\Omega)$. Then $\partial\Omega$ is a boundary for $\mathcal{L}$. Put $E = \partial\Omega \cap U$. With z_n as in (4), put

$$g_n(z) = \frac{1}{z - z_n}.$$

Then $g_n \in \mathcal{L}$. If $\varepsilon > 0$ is small enough, we have for all $x \in \Omega$ with $|x - z_0| < \varepsilon$,

$$|g_n(x)| > \sup_{\partial\Omega \setminus E} |g_n|$$

for all large n. Hence the lemma gives $f(x) = 0$ for all $x \in \Omega$ with $|x - z_0| < \varepsilon$, and so $f \equiv 0$. □

If we do not assume (4), the conclusion follows from

Theorem 10.6 (Radó's Theorem). *Let h be a continuous function on the disk D. Let Z denote the set of zeros of h. If h is analytic on $\mathring{D} \setminus Z$, then h is analytic on $\mathring{D}$.*

Proof. We assume that Z has an empty interior. The case $\mathring{Z} \neq \emptyset$ is treated similarly.

Let $\mathcal{L}$ consist of all sums

$$\sum_{\nu=0}^{N} a_\nu h^\nu, \qquad a_\nu \in A(D).$$

If $f \in \mathcal{L}$, f is analytic in $|z| < 1$ except possibly on Z, so

(5) $$|f(x)| \leq \max_{\Gamma \cup Z} |f|, \qquad \text{all } x \in D.$$

We apply Lemma 10.4 to $\mathcal{L}$ with $X_0 = \Gamma \cup Z$, $E = Z$. Since $h \in \mathcal{L}$ and $h = 0$ on Z we get by the lemma

$$|g(x)| \leq \sup_{\Gamma} |g|, \qquad \text{all } g \in \mathcal{L}, \tag{6}$$

if $x \in D \backslash Z$, since then $h(x) \neq 0$.

By continuity, (6) then holds for all $x \in D$. Thus $\mathcal{L}$ satisfies the hypotheses of Theorem 10.3, and so $\mathcal{L} \subseteq A(D)$. Thus h is analytic on $\mathring{D}$.

Note that Theorem 10.5 follows at once from Radó's theorem.

For future use we next prove

Theorem 10.7. *Let $\mathfrak{A}$ be a uniform algebra on a space X with maximal ideal space $\mathcal{M}$. Let $f \in \mathfrak{A}$ satisfying*
(a) $|f| = 1$ *on* X.
(b) $0 \in f(\mathcal{M})$.
(c) $\exists$ *a closed subset* Γ_0 *of* Γ *having positive linear measure such that for each* $\lambda \in \Gamma_0$ *there is a unique point* q *in* X *with* $f(q) = \lambda$.

Then

(7) *For each* $z_1 \in \mathring{D}$ *there is a unique* x *in* $\mathcal{M}$ *with* $f(x) = z_1$.

(8) *If* $g \in \mathfrak{A}$, $\exists G$ *analytic in* $\mathring{D}$ *such that*

$$g = G(f) \qquad \text{on } f^{-1}\mathring{D}).$$

Proof. For each measure μ on X, let $f(\mu)$ denote the induced measure on Γ; i.e., for $S \subset \Gamma$,

$$f(\mu)(S) = \mu(f^{-1}(S)).$$

where $f^{-1}(S) = \{x \in X | f(x) \in S\}$.

Since by (b), $f(\mathcal{M})$ contains 0, and by (a), $f(X) \subset \Gamma$, it follows that $f(\mathcal{M}) \supset D$. (Why? See Lemma 11.1.) Fix p_1 and p_2 in $\mathcal{M}$ with

$$f(p_1) = f(p_2) = z_1 \in \mathring{D}.$$

We must show that $p_1 = p_2$. Suppose not. Then $\exists g \in \mathfrak{A}$ with $g(p_1) = 1$ and $g(p_2) = 0$. Choose, by Exercise 1.2, positive measures μ_1 and μ_2 on X with

$$h(p_j) = \int h d\mu_j, \qquad \text{all } h \in \mathfrak{A},$$

for $j = 1, 2$.

Let G be a polynomial. Then

$$\int G d(f(\mu_1)) = \int G(f) d\mu_1 = G(f(p_1))$$

and similarly for μ_2. Hence $f(\mu_1) - f(\mu_2)$ is a real measure on Γ annihilating the polynomials. Hence $f(\mu_1) - f(\mu_2) = 0$. (Why?)

Since by (c), f maps $f^{-1}(\Gamma_0)$ bijectively on Γ_0, it follows that μ_1 and μ_2 coincide when restricted to $f^{-1}(\Gamma_0)$. Hence the same holds for the measure $g\mu_1$ and $g\mu_2$.

Put $\lambda_j = f(g\mu_j)$, $j = 1, 2$. Then λ_1 and λ_2 coincide when restricted to Γ_0. For a polynomial G we have

$$\int G d\lambda_j = \int G(f) g d\mu_j = G(f(p_j))g(p_j).$$

Hence by choice of g,

$$\int G d\lambda_1 = G(z_1),$$
$$\int G d\lambda_2 = 0.$$

Thus

$$\int G d(\lambda_1 - \lambda_2) = G(z_1), \qquad \text{all } G. \tag{9}$$

It follows that the measure $(z - z_1)d(\lambda_1 - \lambda_2)$ is orthogonal to all polynomials. By the theorem of F. and M. Riesz (see [Bi2, Chap. 4]), $\exists k \in H^1$ with

$$(z - z_1)\, d(\lambda_1 - \lambda_2) = k\, dz.$$

It follows that $k = 0$ on Γ_0. Since Γ_0 has positive measure, $k \equiv 0$. (See [Hof, Chap. 4].) But $z - z_1 \neq 0$ on Γ, so $\lambda_1 - \lambda_2 = 0$, contradicting (9). Hence $p_1 = p_2$, and (7) is proved.

It follows from (7) that if $g \in \mathfrak{A}$, $\exists G$ continuous on $\mathring{D}$, with $g = G(f)$ on $f^{-1}(\mathring{D})$. It remains to show that G is analytic.

Fix an open disk U with closure $\bar{U} \subset \mathring{D}$. Let $\mathcal{L}$ be the algebra of all functions

$$G = g(f^{-1}), \qquad g \in \mathfrak{A},$$

restricted to $\bar{U}$.

Choose $x \in U$. $f^{-1}(U)$ is an open subset of $\mathcal{M}$ with boundary $f^{-1}(\partial U)$, and $f^{-1}(x) \in f^{-1}(U)$.

By the local maximum modulus principle, if $h \in \mathfrak{A}$,

$$|h(f^{-1}(x))| \leq \max_{f^{-1}(\partial U)} |h|$$

or

$$|H(x)| \leq \max_{\partial U} |H|$$

if $H = h(f^{-1}) \in \mathcal{L}$. Note also that $z = f(f^{-1}) \in \mathcal{L}$.

Theorem 10.3 (which clearly holds if D is replaced by an arbitrary disk) now applies to the algebra $\mathcal{L}$ on $\bar{U}$. We conclude that $\mathcal{L} \subseteq A(\bar{U})$, and so $G = g(f^{-1})$ is analytic in U for every $g \in \mathfrak{A}$.

Thus G is analytic in $\mathring{D}$, whence (8) holds. $\square$

NOTES

Lemma 10.1 and the proof of Theorem 10.2 based on it are due to Paul Cohen, A note on constructive methods in Banach algebras, *Proc. Am. Math. Soc.* **12** (1961). Theorem 10.2 is due to J. Wermer. On algebras of continuous functions, *Proc. Am. Math. Soc.* **4** (1953). Paul Cohen's proof of Theorem 10.2 developed out of an abstract proof of the same result by K. Hoffman and I. M. Singer, Maximal algebras of continuous functions, *Acta Math.* **103** (1960). Theorem 10.3 is due to W. Rudin, Analyticity and the maximum modulus principle, *Duke Math. J.* **20** (1953). Lemma 10.4 is a result of I. Glicksberg, Maximal algebras and a theorem of Radó, *Pacific J. Math.* **14** (1964). Theorem 10.6 is due to T. Radó and has been given many proofs. See, in particular, E. Heinz, Ein elementarer Beweis des Satzes von Radó-Behnke-Stein-Cartan. The proof we have given is to be found in the paper of Glicksberg cited above. Theorem 10.7 is due to E. Bishop and is contained in Lemma 13 of his paper [Bi3].

11

Maximum Modulus Algebras

Let A be an algebra of functions defined and analytic on a plane region Ω. Fix a disk $\Delta = \{z : |z - z_0| \leq r\} \subseteq \Omega$. Then the inequality

$$(1) \qquad |f(z_0)| \leq \max_{z \in \partial\Delta} |f(z)|$$

holds for every function f in A.

In more complicated situations, one often meets the following generalization of (1): We consider an algebra A of continuous complex-valued functions defined on a locally compact Hausdorff space X. We assume that A separates the points of X. We fix a function p in A and an open set Ω in $\mathbb{C}$, such that p is a proper mapping of X onto Ω, "proper" meaning that $p^{-1}(K)$ is compact for each compact set K in Ω. We now assume, for each $\lambda_0 \in \Omega$ and each closed disk Δ centered at λ_0, the inequality

$$(2) \qquad |g(x^0)| \leq \max_{p^{-1}(\partial\Delta)} |g|$$

for each $x^0 \in p^{-1}(\lambda_0)$ and $g \in A$.

If (2) holds, we say that (A, X, Ω, p) is a *maximum modulus algebra* (on X, with projection p over Ω).

EXERCISE 11.1. Let X be the product of the open unit disk and the closed unit interval, i.e.,

$$X = \{(\lambda, t) : \lambda \in \mathbb{C}, |\lambda| < 1, 0 \leq t \leq 1\}.$$

Let A be the algebra of functions continuous on X, such that, for all t, $0 \leq t \leq 1$,

$$\lambda \longmapsto f(\lambda, t)$$

is analytic on $\{|\lambda| < 1\}$. Put $p(\lambda, t) = \lambda$. Show that (A, X, Ω, p) is a maximum modulus algebra on X.

EXERCISE 11.2. Let λ, w be complex coordinates in $\mathbb{C}^2$. Let Σ denote the complex curve $w^2 = z$ in $\mathbb{C}^2$, i.e., $\Sigma = \{(z, w) \in \mathbb{C}^2 : w^2 = z\}$. We take $X = \Sigma \cap (\Omega \times \mathbb{C})$, where $\Omega = \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$ and $p(\lambda, w) = \lambda$. Note that for each $\lambda \in \Omega$, $p^{-1}(\lambda)$ is the pair of points $(\lambda, \sqrt{\lambda})$, $(\lambda, -\sqrt{\lambda})$. Let A be the algebra of all functions g on X that are analytic on X, in the sense that near each point $(\lambda_0, w_0) \in X$, g can be written as $g = G \circ p$ for some G analytic on a neighborhood of λ_0. Show that (A, X, Ω, p) is a maximum modulus algebra on X.

We now fix a plane region Ω that contains the closed unit disk and consider a maximum modulus algebra (A, X, Ω, p) over Ω. We put Γ equal to the unit circle and $Y = p^{-1}(\Gamma)$. We fix a point x^0 in $p^{-1}(0)$. As in Exercise 1.2, this yields the existence of a probability measure μ on Y such that

$$f(x^0) = \int_Y f d\mu$$

for all $f \in A$; μ is a *representing measure* for x^0. Each continuous function ϕ on Γ "pulls back" to a function $\tilde{\phi}$ on Y defined by $\tilde{\phi} = \phi \circ p$. We define the "push forward" μ_* of μ as the measure on Γ given by

$$\mu_*(E) = \mu(p^{-1}(E)) \tag{3}$$

for each Borel set $E \subseteq \Gamma$. For each $\phi \in C(\Gamma)$, we then have

$$\int_Y \tilde{\phi} d\mu = \int_\Gamma \phi d\mu_*, \tag{4}$$

as is easily verified. Also, clearly, μ_* is a probability measure on Γ.

In particular, fixing a positive integer n and putting $\phi(\lambda) = \lambda^n$, $\lambda \in \Gamma$ we get $\tilde{\phi}(y) = p^n(y)$, $y \in Y$, and so (4) now gives

$$\int_Y p^n d\mu = \int_\Gamma \lambda^n d\mu_*, \qquad n = 1, 2, \ldots.$$

By the choice of μ, the left-hand side equals $p^n(x^0) = 0$, since x^0 lies over 0. So

$$0 = \int_\Gamma \lambda^n d\mu_*, \qquad n = 1, 2, \ldots.$$

Taking complex conjugates, we get

$$0 = \int_\Gamma \bar{\lambda}^n d\mu_*, \qquad n = 1, 2, \ldots.$$

Also, $1 = \int_\Gamma d\mu_*$.

Hence the Fourier coefficients of the measures μ_* and $d\theta/2\pi$ coincide, and hence $\mu_* = d\theta/2\pi$. It follows that

$$\int_Y \tilde{\phi} d\mu = \frac{1}{2\pi} \int_\Gamma \phi d\theta, \qquad \phi \in C(\Gamma). \tag{5}$$

Hence, if $\phi \in C(\Gamma)$, we deduce that

$$\int_Y |\tilde{\phi}|^2 d\mu = \frac{1}{2\pi} \int_\Gamma |\phi|^2 d\theta. \tag{6}$$

We now form the space $L^2(\mu)$ of all functions on Y measurable-$d\mu$ and square summable. Fix $\phi \in L^2(\Gamma, \frac{d\theta}{2\pi})$. We shall "lift" ϕ to a function $\tilde{\phi}$ on Y as follows: Choose a sequence $\{\phi_n\} \in C(\Gamma)$ such that $\phi_n \to \phi$ in $L^2(\Gamma, \frac{d\theta}{2\pi})$. In view of (6), the sequence $\tilde{\phi_n}$ converges in $L^2(\mu)$. We put $\tilde{\phi} = \lim_{n\to\infty} \tilde{\phi_n}$, in $L^2(\mu)$. Again by (6), $\tilde{\phi}$ is independent of the choice of the sequence $\{\phi_n\}$, and (6) remains valid for ϕ and $\tilde{\phi}$. We define a subspace $\mathcal{C}$ of $L^2(\mu)$ by,

$$\mathcal{C} = \{\tilde{\phi} \in L^2(\mu) : \phi \in L^2(\Gamma, \frac{d\theta}{2\pi})\}.$$

$\mathcal{C}$ is then a closed subspace of $L^2(\mu)$. We regard its elements as those functions in $L^2(\mu)$ which are "constant on each fiber of the map p." We may identify $\mathcal{C}$ with $L^2(\Gamma, \frac{d\theta}{2\pi})$ by identifying $\tilde{\phi}$ with ϕ.

We now shall consider the following: We fix a function F in A. Restricted to Y, F lies in $L^2(\mu)$. We shall study the orthogonal projection G of F on the subspace $\mathcal{C}$ and show that G has interesting properties related to F. We write $co(S)$ for the closed convex hull of a set $S \subseteq \mathbb{C}$.

Theorem 11.1. *Fix $F \in A$. Let G denote the orthogonal projection in $L^2(\mu)$ of F to $\mathcal{C}$. Then*

(7) $$G \in H^\infty \text{ on } \Gamma \qquad \text{and}$$

(8) $$G(0) = F(x^0)$$

(9) $$\text{For a.a. } \theta \in [0, 2\pi],\ G(e^{i\theta}) \in co(F(p^{-1}(e^{i\theta}))).$$

Note. H^∞ denotes the space of functions in $L^\infty(\Gamma)$ that are a.e. radial limits of functions bounded and analytic in the unit disk.

Proof. $F - G$ is orthogonal to $\mathcal{C}$ in $L^2(\mu)$, or

$$\int_Y (F - G)\bar{g}\,d\mu = 0, \qquad \forall g \in \mathcal{C}.$$

This is equivalent to

(10) $$\int_Y F\bar{g}\,d\mu = \int_Y G\bar{g}\,d\mu.$$

Since G and $\bar{g} \in \mathcal{C}$, using our identification of $\mathcal{C}$ with $L^2(\Gamma, \frac{d\theta}{2\pi})$, G and $\bar{g} \in L^2(\Gamma, \frac{d\theta}{2\pi})$, and we verify that the right side in (10) is $(1/2\pi)\int_\Gamma G(e^{i\theta})\overline{g(e^{i\theta})}d\theta$; so we get

(11) $$\int_Y F\bar{g}\,d\mu = \frac{1}{2\pi}\int_\Gamma G\bar{g}\,d\theta.$$

We now fix a positive integer n and put $g = \bar{p}^n$. Then g is identified with $\bar{\lambda}^n$ in $L^2(\Gamma, \frac{d\theta}{2\pi})$, and so we get

$$\int_Y F p^n d\mu = \frac{1}{2\pi} \int_\Gamma G \lambda^n d\theta. \tag{12}$$

By choice of μ and choice of x^0, the left-hand side $= F(x^0)(p(x^0))^n$. So

$$0 = \int_\Gamma G(e^{i\theta}) e^{in\theta} d\theta, \qquad n = 1, 2, \ldots. \tag{13}$$

Thus G belongs to the Hardy space H^2 on Γ.

Taking $g = 1$ in (11), we get $F(x^0) = \int_Y F d\mu = \frac{1}{2\pi} \int_\Gamma G d\theta = G(0)$, and so assertion (8) holds.

Fix next $\lambda_0 = e^{i\theta_0} \in \Gamma$. For each $\delta > 0$, let α denote the arc of Γ from $e^{i(\theta_0 - \delta)}$ to $e^{i(\theta_0+\delta)}$ and put

$$g_\delta(e^{i\theta}) = \begin{cases} \pi/\delta, & \theta_0 - \delta \le \theta \le \theta_0 + \delta \\ 0, & |\theta - \theta_0| > \delta \end{cases}$$

so that

$$\int_Y F \bar{g}_\delta d\mu = \frac{1}{2\pi} \int_\Gamma G \bar{g}_\delta d\theta = \frac{1}{2\delta} \int_{\theta_0 - \delta}^{\theta_0 + \delta} G(e^{i\theta}) d\theta. \tag{14}$$

Also, $\int_Y \bar{g}_\delta \circ p d\mu = \frac{1}{2\delta} \int_{\theta_0-\delta}^{\theta_0+\delta} \frac{\pi}{\delta} d\theta = 1$, and $\operatorname{supp}(\bar{g}_\delta \circ p d\mu) \subseteq p^{-1}(\alpha)$. The right side of (14) approaches $G(\lambda_0)$ for a.a. $\lambda_0 \in \Gamma$. The left side of (14) approaches a point in the convex hull of the set $F(p^{-1}(\lambda_0))$, since the probability measures $\bar{g}_\delta \circ p d\mu$ have a weak-* convergent subsequence approaching some probability measure supported in $p^{-1}(\lambda_0)$. Hence $G(\lambda_0) \in co(F(p^{-1}(\lambda_0)))$ for a.a. $\lambda_0 \in \Gamma$. Thus assertion (9) holds. Since F is bounded on Y, it follows that $G \in L^\infty(\Gamma)$, and, since $G \in H^2$, this yields $G \in H^\infty$, i.e., (7) holds. Theorem 11.1 is proved. □

We next replace the unit disk by an arbitrary disk Δ with center λ_0, fix a point x_0 in the fiber $p^{-1}(\lambda_0)$, and prove the analogue of Theorem 11.1.

Theorem 11.2. *Let (A, X, Ω, p) be a maximum modulus algebra on Ω and fix $F \in A$. Choose a closed disk Δ contained in Ω, with center λ_0, and fix a point x^0 in $p^{-1}(\lambda_0)$. Then there exists a bounded analytic function G on* int(Δ) *such that*

$$G(\lambda_0) = F(x^0), \text{ and} \tag{15}$$

$$G(\lambda) \in co(F(p^{-1}(\lambda))) \text{ for a.a. } \lambda \in \partial\Delta. \tag{16}$$

Proof. Let $\chi(\lambda) = a\lambda + b$, $a, b \in \mathbb{C}$, be a conformal map of Δ onto the unit disk $\{|z| \le 1\}$, $\chi(\lambda_0) = 0$. Put $\Pi = \chi \circ p$. Then $\Pi \in A$ and χ maps Ω on the region $\chi(\Omega)$ and Δ on the closed unit disk. Hence $(A, X, \chi(\Omega), \Pi)$ is a maximum modulus algebra over $\chi(\Omega)$. Also, $\Pi(x^0) = 0$. By Theorem 11.1 there exists a function $H \in H^\infty$ such that $H(0) = F(x^0)$ and $H(e^{i\theta}) \in co(F(p^{-1}(e^{i\theta})))$,

θ a.e. on Γ. Put $G = H \circ \chi$. Then $G \in H^\infty(\Delta)$, $G(\lambda_0) = F(x^0)$, and $G(\lambda) = H(\chi(\lambda)) \in co(F(\Pi^{-1}(\chi(\lambda)))$ a.e. on $\partial\Delta$. Also, $\Pi^{-1}(\chi(\lambda)) = p^{-1}(\lambda)$, so $G(\lambda) \in co(F(p^{-1}(\lambda)))$ a.e. on $\partial\Delta$. This gives the theorem. □

Corollary 11.3. *Let (A, X, Ω, f) be a maximum modulus algebra and fix a closed disk $\Delta \subseteq \Omega$. Assume that X lies one sheeted over Δ, in the sense that $f^{-1}(\lambda)$ consists of a single point for each $\lambda \in \Delta$. Then, over Δ, every $g \in A$ is an analytic function of f, i.e., there exists G analytic on intΔ and continuous on Δ such that $g = G \circ f$ on $f^{-1}(\Delta)$.*

Proof. Fix $g \in A$. By Theorem 11.2, there exists a bounded analytic function G on int(Δ) such that

$$G(\lambda) \in co(g(f^{-1}(\lambda)) \text{ for a.a. } \lambda \in \partial\Delta.$$

By hypothesis, $f^{-1}(\lambda)$ is a singleton; so $co(g(f^{-1}(\lambda))) = g(f^{-1}(\lambda))$, and so $G(\lambda) = g(f^{-1}(\lambda))$ a.e. on $\partial\Delta$. f is a one-one continuous map of $f^{-1}(\partial\Delta)$ onto $\partial\Delta$, and therefore f^{-1} is continuous on $\partial\Delta$, and therefore $g(f^{-1}(\lambda))$ is continuous on $\partial\Delta$.

It follows that G is continuous on the closed disk Δ. Hence we can choose a sequence of polynomials $\{P_n\}$ such that $P_n \to G$ uniformly on Δ. Hence $P_n \circ f \to G \circ f$ uniformly on $f^{-1}(\partial\Delta)$. $P_n \circ f \in A$, for each n, and tends to $G \circ f = g$ uniformly on $f^{-1}(\partial\Delta)$. By the maximum principle for A, it follows that, on $f^{-1}(\Delta)$,

$$|P_n \circ f - g| \leq \max_{f^{-1}(\partial\Delta)} |P_n \circ f - g|.$$

Hence $|P_n \circ f - g| \to 0$ uniformly on $f^{-1}(\Delta)$, and $G \circ f = g$ on $f^{-1}(\Delta)$. □

We fix a maximum modulus algebra (A, X, Ω, p). To each $F \in A$, various scalar-valued functions defined on Ω are associated, by considering for each λ the set

$$F(p^{-1}(\lambda)) \subset \mathbb{C},$$

i.e., the image under F of the fiber over λ. Each such set is compact.

Definition 11.1. $Z_F(\lambda) = \max_{y \in p^{-1}(\lambda)} |F(y)|$, $\lambda \in \Omega$.

Definition 11.2. Fix an integer $n \geq 2$. Let S be a compact set contained in $\mathbb{C}$. Put

$$d_n(S) = \max_{z_1, z_2, \cdots, z_n \in S} \left(\prod_{j<k} |z_j - z_k|\right)^{\frac{2}{n(n-1)}}.$$

We call $d_n(S)$ the n**-diameter of** S.

Example.

$$d_2(S) = \max_{z_1, z_2 \in S} |z_1 - z_2|.$$

So $d_2(S)$ is just the diameter of S:

$$d_3(S) = \max_{z_1, z_2, z_3 \in S} (|z_1 - z_2||z_1 - z_3||z_2 - z_3|)^{\frac{1}{3}}.$$

For $F \in A$ and n fixed, the function

$$\lambda \mapsto d_n[F(p^{-1}(\lambda))], \qquad \lambda \in \Omega$$

is another example of a scalar-valued function defined on Ω, attached to F.

EXERCISE 11.3. Fix $F \in A$. Then Z_F is upper semicontinuous on Ω; i.e., for each $\lambda_0 \in \Omega$,

$$Z_F(\lambda_0) \geq \limsup_{\lambda \to \lambda_0} Z_F(\lambda).$$

A real-valued function defined on Ω is *subharmonic* on Ω if:

(i) u is upper semicontinuous at each $\lambda \in \Omega$, and

(ii) For each closed disk $\Delta = \{|\lambda - \lambda_0| \leq r\}$ contained in Ω, we have the inequality

$$u(\lambda_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(\lambda_0 + re^{i\theta}) d\theta. \tag{17}$$

See Appendix A1 for references to subharmonic functions.

Theorem 11.3. *Let (A, X, Ω, p) be a maximum modulus algebra over Ω. Fix $F \in A$. Then $\lambda \mapsto \log Z_F(\lambda)$ is subharmonic on Ω.*

PROOF. In view of Exercise 11.3, it suffices to show that $\log Z_F$ satisfies the inequality (17).

We fix a disk $\Delta = \{|\lambda - \lambda_0| \leq r\}$ contained in Ω and apply Theorem 11.2 to the function F, a point $x^0 \in p^{-1}(\lambda_0)$, and the disk Δ. This yields $G \in H^\infty(\text{int}\Delta)$ with $G(\lambda_0) = F(x^0)$, and, by (16), $|G(\lambda)| \leq \max_{y \in p^{-1}(\lambda)} |F(y)| = Z_F(\lambda)$ for a.a. $\lambda \in \partial\Delta$. By Jensen's inequality on $\text{int}\Delta$, we have

$$\log |G(\lambda_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |G(\lambda_0 + re^{i\theta})| d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \log Z_F(\lambda_0 + re^{i\theta}) d\theta.$$

The left-hand side $= \log |F(x^0)|$, so inequality (17) holds, and we are done. $\square$

We wish to study the functions

$$\lambda \mapsto \log d_n[F(p^{-1}(\lambda))], \qquad \lambda \in \Omega,$$

with d_n given by Definition 11.2.

We next develop some machinery concerning the n-fold tensor product of the algebra A. Fix an integer $n \geq 1$. Define $X^n = X \times X \times \cdots \times X$, the topological product of n copies of X. Define the n-fold tensor product of A, $\otimes^n A$, by

$$\otimes^n A = \{g \in C(X^n) : g(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{N} g_{j1}(x_1) g_{j2}(x_2) \ldots g_{jn}(x_n),$$

$(x_1, x_2, \ldots, x_n) \in X^n$, where each $g_{j\nu} \in A$, N is arbitrary}.

Define the map $\Pi : X^n \to \mathbb{C}^n$ by $\Pi(x_1, x_2, \cdots, x_n) = (p(x_1), p(x_2), \cdots, p(x_n))$. Let Δ^n denote the closed polydisk in $\mathbb{C}^n$ and $T^n = \{(z_1, z_2, \cdots, z_n) \in \Delta^n : |z_j| = 1, 1 \le j \le n\}$. Put $\gamma = \{(\lambda, \lambda, \cdots, \lambda) \in T^n : |\lambda| = 1\}$, so γ is a closed curve lying in T^n.

Theorem 11.4. *Let (A, X, Ω, p) be a maximum modulus algebra over Ω. Assume that $\Delta = \{|z| \le 1\} \subset \Omega$. Fix $F \in \otimes^n A$ and fix $x^0 \in \Pi^{-1}(0, 0, \cdots, 0)$. Then*

$$|F(x^0)| \le \max_{\Pi^{-1}(\gamma)} |F|.$$

Note. By definition,

$$\Pi^{-1}(\gamma) = \{(x_1, x_2, \cdots, x_n) \in X^n : p(x_1) = \cdots = p(x_n), \qquad \text{and } |p(x_1)| = 1\}$$

To prove Theorem 11.4, we need the following.

Lemma 11.5. *Under the hypothesis of Theorem 11.4, there exists a function $G \in H^\infty(T^n)$ such that*

$$G(0, 0, \cdots, 0) = F(x^0), \ \textit{and,} \tag{18}$$

if $\mathcal{U}$ is any relatively open subset of T^n,

$$||G||_{L^\infty(\mathcal{U})} \le \sup_{\Pi^{-1}(\mathcal{U})} |F|. \tag{19}$$

Remark. Just as in the case where $n = 1$, $H^\infty(T^n)$ is defined as the set $G \in L^\infty(T^n, d\theta_1 \cdots d\theta_n)$ such that $\int_{T^n} Ge^{is_1\theta_1} \ldots e^{is_n\theta_n} d\theta_1 \ldots d\theta_n = 0$, if $s_1, \cdots, s_n \in \mathbb{Z}$ and $s_j > 0$ for some j. An analogous definition gives $H^2(T^n)$. $H^\infty(T^n)$ can be identified, by the Cauchy integral formula, with the Banach algebra of bounded analytic functions in the open unit polydisk. Thus, in (18), $G(0, \cdots, 0)$ denotes the value at $0 \in \mathbb{C}^n$ of the extension of $G \in H^\infty(T^n)$ to the polydisk.

PROOF. $x^0 = (x_1^0, \cdots, x_n^0)$ and $p(x_j^0) = 0$, for all j. We may choose representing measures μ_k for x_k^0, supported on $p^{-1}(\partial\Delta)$, such that

$$g(x_k^0) = \int_{p^{-1}(\partial\Delta)} g d\mu_k, \qquad g \in A, \text{ for } k = 1, 2, \ldots, n.$$

We form the product measure $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$, supported on $\Pi^{-1}(T^n)$. We denote by μ_* the "push forward" of μ on T^n under the map Π.

EXERCISE 11.4. Show that the measure

$$\mu_* = \left(\frac{1}{2\pi}\right)^n d\theta_1 d\theta_2 \cdots d\theta_n, \text{ on } T^n.$$

We claim that μ is a representing measure, for the algebra $\otimes^n A$, i.e.,

$$\int_{\Pi^{-1}(T^n)} h d\mu = h(x^0), \qquad h \in \otimes^n A. \tag{20}$$

Without loss of generality, we consider $h(x) = g_1(x_1)g_2(x_2)\cdots g_n(x_n)$, $x = (x_1, x_2, \cdots, x_n)$, $g_j \in A$ for all j:

$$\begin{aligned}
&\int_{\Pi^{-1}(T^n)} h d\mu \\
&\quad = \int_{\Pi_{j=1}^n (p_j^{-1}(\partial\Delta))} g_1(x_1)g_2(x_2)\cdots g_n(x_n) d\mu_1(x_1) \\
&\qquad \times d\mu_2(x_2) \times \cdots \times d\mu_n(x_n) \\
&= \Pi_{j=1}^n \int_{p_j^{-1}(\partial\Delta)} g_j(x_j) d\mu_j(x_j) = \Pi_{j=1}^n g_j(x_j^0) = h(x^0),
\end{aligned}$$

proving (20).

For $\phi \in C(T^n)$, we define $\tilde{\phi}$ on $\Pi^{-1}(T^n)$ by $\tilde{\phi}(y) = \phi(\Pi(y))$, $y \in \Pi^{-1}(T^n)$. Then we have

$$\int_{\Pi^{-1}(T^n)} \tilde{\phi} d\mu = \int_{T^n} \phi d\mu_* = \left(\frac{1}{2\pi}\right)^n \int_{T^n} \phi d\theta_1 \cdots d\theta_n, \tag{21}$$

and so

$$\int_{\Pi^{-1}(T^n)} |\tilde{\phi}|^2 d\mu = \left(\frac{1}{2\pi}\right)^n \int_{T^n} |\phi|^2 d\theta_1 \cdots d\theta_n. \tag{22}$$

This last inequality allows us to lift each $\phi \in L^2(T^n)$, taken with respect to the Haar measure on T^n, to a function $\tilde{\phi}$ in $L^2(\mu)$ that is "constant on the fibers of Π." We put, as in Theorem 11.1,

$$\mathcal{C} = \{\tilde{\phi} \in L^2(\mu) : \phi \in L^2(T^n)\}.$$

We identify $\mathcal{C}$ with $L^2(T^n)$, by identifying ϕ with $\tilde{\phi}$. Arguing as in the proof of Theorem 11.1, we assign to a given $F_0 \in \otimes^n A$ the orthogonal projection of F_0 to $\mathcal{C}$, in $L^2(\mu)$, denoted by G_0. Then

$$\begin{aligned}
&\int_{\Pi^{-1}(T^n)} F_0 \bar{g} d\mu \\
&\quad = \int_{\Pi^{-1}(T^n)} G_0 \bar{g} d\mu = \left(\frac{1}{2\pi}\right)^n \int_{T^n} G_0 \bar{g} d\theta_1 \cdots d\theta_n, \qquad g \in \mathcal{C}.
\end{aligned} \tag{23}$$

We claim that $G_0 \in H^2(T^n)$, which means that

$$\int_{T^n} G_0 e^{is_1\theta_1} \ldots e^{is_n\theta_n} = 0, \tag{24}$$

where s_j are integers with at least one of them being positive. Fix such an n-tuple $(s_1, s_2, \cdots, s_n)$, where, without loss of generality, $s_1 > 0$, and put $g = \overline{p_1^{s_1} \cdots p_n^{s_n}}$.

Then, by (23), we have

$$(25) \qquad \int_{\Pi^{-1}(T^n)} F_0 p_1^{s_1} \cdots p_n^{s_n} d\mu = \left(\frac{1}{2\pi}\right)^n \int_{T^n} G_0 e^{is_1\theta_1} \ldots e^{is_n\theta_n} d\theta_1 \ldots d\theta_n,$$

since $p_1^{s_1} \cdots p_n^{s_n}$, in $\mathcal{C}$, is identified with $e^{is_1\theta_1} \ldots e^{is_n\theta_n}$ in $L^2(T^n)$. Without loss of generality, $F_0(x) = g_1(x_1) \ldots g_n(x_n)$, where each $g_j \in A$, and then

$$\begin{aligned}
&\int_{\Pi^{-1}(T^n)} F_0 p_1^{s_1} \ldots p_n^{s_n} d\mu \\
&\quad = \int_{\Pi^{-1}(T^n)} p_2^{s_2} \ldots p_n^{s_n} [p_1^{s_1} g_1 \ldots g_n] d\mu \\
&\quad = \int_{\Pi_{j=2}^n p^{-1}(\partial\Delta)} p_2^{s_2} \ldots p_n^{s_n} \times \Bigg[\int_{p^{-1}(\partial\Delta)} p_1^{s_1}(x_1) g_1(x_1) \ldots \\
&\qquad\qquad g_n(x_n) d\mu_1(x_1) \Bigg] d\mu_2(x_2) \ldots d\mu_n(x_n).
\end{aligned}$$

The inner integral equals $p_1^{s_1}(x_1^0) g_1(x_1^0) g_2(x_2) \ldots g_n(x_n) = 0$, since $p_1(x_1^0) = 0$. By (25), it follows that $\int_{T^n} G_0 e^{is_1\theta_1} \ldots e^{is_n\theta_n} d\theta_1 \cdots d\theta_n = 0$, i.e., (24) holds, and so $G_0 \in H^2(T^n)$, as claimed.

To prove assertion (19), we fix a relatively open subset $\mathcal{U}$ of T^n and put $M = \sup |F|$ on $\Pi^{-1}(\mathcal{U})$. Fix $\lambda^0 = (e^{i\theta_1^0}, e^{i\theta_2^0}, \cdots, e^{i\theta_n^0})$ in $\mathcal{U}$. For each $\delta > 0$, form the set $K_\delta = \{(e^{i\theta_1^0}, e^{i\theta_2^0}, \cdots, e^{i\theta_n^0}) : |\theta_j - \theta_j^0| \leq \delta, 1 \leq j \leq n\}$. We choose g_δ to be the characteristic function of K_δ normalized so that $(\frac{1}{2\pi})^n \int_{T^n} g_\delta d\theta_1 \cdots d\theta_n = 1$. Applying (23) with $g = g_\delta$, $F_0 = F$, and $G_0 = G$, we get

$$(26) \qquad |\int_{\Pi^{-1}(K_\delta)} F g_\delta d\mu| = |c_\delta \int_{K_\delta} G d\theta_1 \ldots d\theta_n|,$$

where $c_\delta = (\text{volume}(K_\delta))^{-1}$. As $\delta \to 0$, the right-hand side tends to $G(\lambda^0)$ for a.a. λ^0 in $\mathcal{U}$. For fixed λ^0 in $\mathcal{U}$ and all small δ, the left-hand side $\leq \max_{\Pi^{-1}(K_\delta)} |F| \leq \sup_{\Pi^{-1}(\mathcal{U})} |F|$. Hence $|G(\lambda^0)| \leq \sup_{\Pi^{-1}(\mathcal{U})} |F|$, a.e. on $\mathcal{U}$. Hence (19) holds and also $G \in H^\infty(T^n)$. The lemma follows. $\square$

Proof of Theorem 11.4. We shall use the fact that, at almost every point, $(e^{i\theta_1}, e^{i\theta_2}, \cdots, e^{i\theta_n}) \in T^n$ relative to the Haar measure $G(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) = \lim_{r\to 1} G(re^{i\theta_1}, re^{i\theta_2}, \cdots, re^{i\theta_n})$. We fix a neighborhood $\mathcal{U}$ of γ on T^n. Fix a circle $\gamma^1 = \{(\zeta e^{i\theta_1}, \zeta e^{i\theta_2}, \cdots, \zeta e^{i\theta_n}) : |\zeta| = 1\} \subseteq \mathcal{U}$ such that for a.a. ζ on $\{|\zeta| = 1\}$, $G(\zeta e^{i\theta_1}, \zeta e^{i\theta_2}, \ldots, \zeta e^{i\theta_n}) = \lim_{r\to 1} G(r\zeta e^{i\theta_1}, r\zeta e^{i\theta_2}, \ldots, r\zeta e^{i\theta_n})$. On the disk bounded by γ^1, that is, on $\{(\zeta e^{i\theta_1}, \zeta e^{i\theta_2}, \cdots, \zeta e^{i\theta_n}) : |\zeta| < 1\}$,

$$|G| \leq \operatorname*{ess\ sup}_{|\zeta|=1} |G(\zeta e^{i\theta_1}, \zeta e^{i\theta_2}, \ldots, \zeta e^{i\theta_n})| \leq \operatorname*{ess\ sup}_{\mathcal{U}} |G| \leq \sup_{\Pi^{-1}(\mathcal{U})} |F|$$

by (19). In particular, $|G(0, 0, \ldots, 0)| \leq \sup_{\Pi^{-1}(\mathcal{U})} |F|$.

Given $\epsilon > 0$, we choose $\mathcal{U}$ such that $\sup_{\Pi^{-1}(\mathcal{U})} |F| \leq \sup_{\Pi^{-1}(\gamma)} |F| + \epsilon$. It follows that

$$|G(0, 0, \dots, 0)| \leq \sup_{\Pi^{-1}(\gamma)} |F| + \epsilon.$$

Since ϵ is arbitrary, we conclude that

$$|G(0, 0, \dots, 0)| \leq \sup_{\Pi^{-1}(\gamma)} |F|.$$

Also, by (18), $G(0, 0, \dots, 0) = F(x^0)$. So

$$|F(x^0)| \leq \sup_{\Pi^{-1}(\gamma)} |F|.$$

Theorem 11.4 is proved. □

We now look at the "diagonal" of the product space X^n, given by

Definition 11.3.

$$X^{(n)} = \{(x_1, x_2, \dots, x_n) \in X^n : p(x_1) = p(x_2) = \cdots = p(x_n)\}.$$

We define the projection function $\pi : X^{(n)} \to \Omega$ by

$$\pi(x_1, x_2, \dots, x_n) = p(x_1)(= p(x_2) = \cdots = p(x_n)).$$

We also put

Definition 11.4. $A^{(n)}$ is the restriction of $\otimes^n A$ to $X^{(n)}$.

Theorem 11.6. $(A^{(n)}, X^{(n)}, \Omega, \pi)$ *is a maximum modulus algebra.*

PROOF. Clearly, $A^{(n)}$ is an algebra of continuous functions on $X^{(n)}$. Let Δ be a closed disk in Ω with center λ_0. Fix $x^0 \in \pi^{-1}(\lambda_0) \subseteq X^{(n)}$. We must show that

$$|F(x^0)| \leq \max_{\pi^{-1}(\partial\Delta)} |F|, \qquad F \in \otimes^n A. \tag{27}$$

Without loss of generality, $\lambda_0 = 0$ and Δ is the unit disk. We have that $\pi^{-1}(\partial\Delta) = \{(x_1, x_2, \dots, x_n) : p(x_1) = \cdots = p(x_n) = \zeta \text{ with } |\zeta| = 1\}$ and $\Pi^{-1}(\gamma) = \{(x_1, x_2, \dots, x_n) : (p(x_1), \dots, p(x_n)) \in \gamma\}$. Since $\gamma = \{(\lambda, \lambda, \dots, \lambda) : |\lambda| = 1\}$, $\pi^{-1}(\partial\Delta) = \Pi^{-1}(\gamma)$. By Theorem 11.4, $|F(x^0)| \leq \max_{\Pi^{-1}(\gamma)} |F|$. Hence (27) holds and Theorem 11.6 is proved. □

Now let (A, X, Ω, f) be a maximum modulus algebra. Fix $g \in A$. For each $\lambda \in \Omega$, the set $g(f^{-1}(\lambda)) \subseteq \mathbb{C}$. We fix an integer n and we form the function

$$\lambda \mapsto d_n(g(f^{-1}(\lambda))), \qquad \lambda \in \Omega,$$

where d_n is the n-diameter defined in Definition 11.2.

Theorem 11.7. $\lambda \mapsto \log d_n(g(f^{-1}(\lambda)))$ *is subharmonic on* Ω.

PROOF. For ease of understanding we take $n = 3$. The proof is the same for each $n \geq 2$. $\log(d_3(g(f^{-1}(\lambda)))) = \log[\max |z_1 - z_2||z_1 - z_3||z_2 - z_3|]^{1/3}$, the maximum being taken over all triples (z_1, z_2, z_3) with each $z_j \in g(f^{-1}(\lambda))$.

Now the statement $z_j \in g(f^{-1}(\lambda))$ means that there exists $x_j \in f^{-1}(\lambda)$ with $g(x_j) = z_j$. So a triple (z_1, z_2, z_3) is in the competition exactly when it has the form $(g(x_1), g(x_2), g(x_3))$ with $f(x_j) = \lambda$, $j = 1, 2, 3$. By Definition 11.3, the triple $(x_1, x_2, x_3) \in X^{(3)}$ if and only if $f(x_1) = f(x_2) = f(x_3)$ and $\pi(x_1, x_2, x_3) = f(x_1)$, where π is the projection of $X^{(3)}$ to Ω. Hence

$$\log d_3(g(f^{-1}(\lambda)))$$
$$= \log[\max_{\pi(x_1,x_2,x_3)=\lambda} |g(x_1) - g(x_2)||g(x_1) - g(x_3)||g(x_2) - g(x_3)|]^{1/3}.$$

We define, for $(x_1, x_2, x_3) \in X^{(3)}$,

$$G(x_1, x_2, x_3) = (g(x_1) - g(x_2))(g(x_1) - g(x_3))(g(x_2) - g(x_3)).$$

Thus $G \in A^{(3)}$. We have $\log d_3(g(f^{-1}(\lambda))) = \frac{1}{3}\log[\max |G(x_1, x_2, x_3)|$, the maximum being taken over $\pi^{-1}(\lambda) \subseteq X^{(3)}$.

By Theorem 11.6, $(A^{(3)}, X^{(3)}, \Omega, \pi)$ is a maximum modulus algebra, and, hence, by Theorem 11.3, $\lambda \mapsto \log[\max_{\pi^{-1}(\lambda)} |G|]$ is subharmonic on Ω. Thus $\lambda \mapsto \log(d_3(g(f^{-1}(\lambda))))$ is subharmonic on Ω and Theorem 11.7 is proved. $\square$

We shall write $\#S$ for the cardinality of a set S.

Definition 11.5. Let (A, X, Ω, p) be a maximum modulus algebra. Let E be a subset of Ω. For n an integer ≥ 1, we say that (A, X, Ω, p) lies at most n-sheeted over E if $\#p^{-1}(\lambda) \leq n$ for each $\lambda \in E$.

EXAMPLE 11.3. Let Ω be a region in $\mathbb{C}$ and let $a_1, a_2, \ldots, a_n$ be analytic functions defined on Ω. We let X be the set in $\mathbb{C}^2$ defined by the equation

$$w^n + a_1(z)w^{n-1} + a_2(z)w^{n-2} + \cdots + a_n(z) = 0, \tag{28}$$

in the sense that $X = \{(z, w) \in \mathbb{C}^2 : (z, w) \text{ satisfies } (28)\}$.

Let A be the algebra consisting of all restrictions to X of polynomials in z and w. Then A is an algebra of continuous functions on X. Put $p(z, w) = z$, for $(z, w) \in X$. Then $p \in A$ and $p : X \to \Omega$ is a proper map.

We claim that (A, X, Ω, p) is a maximum modulus algebra, and that it lies at most n-sheeted over Ω—provided that the polynomial of (28) satisfies an additional hypothesis on its discriminant, to be formulated below.

For each $z \in \Omega$, equation (28) has n roots in $\mathbb{C}$, and we denote these roots by $w_1(z), w_2(z), \ldots, w_n(z)$, taken in some order. If σ is any symmetric function of n variables, the number $\sigma(w_1(z), w_2(z), \ldots, w_n(z))$ is independent of the order of the roots and hence gives a single-valued function of z on Ω.

In particular, if we take σ to be $\prod_{i<j}(w_i - w_j)^2$, and define $D(z) = \prod_{i<j}(w_i(z) - w_j(z))^2$, then D is a single-valued function on Ω called the discriminant.

HYPOTHESIS. We shall assume that D is not identically 0 on Ω.

The coefficient functions a_j in (28) correspond to the elementary symmetric functions. Since $\prod_{i<j}(w_i - w_j)^2$ is a polynomial in the elementary symmetric functions, it follows that D is analytic in Ω. Since, by hypothesis, D is not identically 0, the zeros of D form a discrete subset Λ of Ω; Λ is empty, finite, or countably infinite.

Fix $z_0 \in \Omega \setminus \Lambda$. Then the roots $w_1(z), w_2(z), \ldots, w_n(z)$ are distinct. Cauchy theory yields that, in some neighborhood $\mathcal{U}$ of z_0, there are n single-valued analytic functions $w_1, w_2, \ldots, w_n$ that provide the roots of (28). For $z \in \mathcal{U}$, the points of X over z, i.e., which are mapped to z by p, are the points $(z, w_1(z)), (z, w_2(z)), \ldots, (z, w_n(z))$. Fix a function $f \in A$. Then there exists a polynomial $Q(z, w)$ such that $f((z, w_j(z))) = Q(z, w_j(z))$, $j = 1, \ldots, n$. Thus the function $z \mapsto f((z, w_j(z)))$ is analytic on $\mathcal{U}$ for each j. We define the symmetric function $\sigma(\alpha_1, \alpha_2, \cdots, \alpha_n) = \max_{1\le j\le n} |\alpha_j|$. Hence the function $u : z \mapsto \max_{1\le j\le n} |f((z, w_j(z)))|$ is well defined on $\Omega \setminus \Lambda$. By the above discussion, $|f((z, w_j(z)))|$ is locally subharmonic at each point z_0 in $\Omega \setminus \Lambda$. Hence u is subharmonic on $\Omega \setminus \Lambda$ and has isolated singularities at the points of Λ. In a deleted neighborhood of each point of Λ, u is locally bounded. It follows (see [Tsu] Thm. III.30) that, if we define $u(z_1) = \overline{\lim}_{z\to z_1} u(z)$, then u is subharmonic on all Ω. We claim that the equality $u(z) = \max_{1\le j\le n} |f((z, w_j(z)))|$ remains true at points $z \in \Lambda$; by the definition of u, we already know that it holds for points $z \in \Omega \setminus \Lambda$. Let $\lambda \in \Lambda$ and fix one of the roots $w_{j_0}(\lambda)$ at λ. Then, by the Cauchy theory, there exists $z_k \to \lambda$, $z_k \in \Omega \setminus \Lambda$ and points $\{w_{j_k}(z_k)\}$ such that $w_{j_k}(z_k) \to w_{j_0}(\lambda)$. It follows that $z \mapsto \max_{1\le j\le n} |f((z, w_j(z)))|$ is continuous at $z = \lambda$ and so the claim follows.

Now let Δ be any closed disk contained in Ω, with center λ_0, and let x_0 be a point of X lying over λ_0. Since x_0 can be written $(\lambda_0, w_j(\lambda_0))$ for some j, $|f(x_0)| \le \max_{1\le j\le n} |f(\lambda_0, w_j(\lambda_0))| = u(\lambda_0)$. Since u is subharmonic on Ω, $u(\lambda_0) \le \max_{\lambda\in\partial\Delta} u(\lambda) = \max_{p^{-1}(\Delta)}(|f|)$. Hence $|f(x_0)| \le \max_{p^{-1}(\Delta)}(|f|)$. Thus (A, X, Ω, f) is a maximum modulus algebra, as claimed. That it lies at most n-sheeted over Ω is clear from the definition. □

We next show that a maximum modulus algebra (A, X, Ω, f) which lies finite-sheeted over a sufficiently large subset E of Ω is an algebra of analytic functions on a certain Riemann surface, in the sense of the following theorem. See the Appendix for the notion of logarithmic capacity.

Theorem 11.8. *Let (A, X, Ω, f) be a maximum modulus algebra. Assume that, for some integer n, there exists a Borel set $E \subseteq \Omega$ of logarithmic capacity $c(E) > 0$, such that, for every $\lambda \in E$, $\#f^{-1}(\lambda) \le n$. Then:*

(i) *$\#f^{-1}(\lambda) \le n$ for every $\lambda \in \Omega$, and*

(ii) *there exists a discrete subset* Λ *of* Ω *such that* $f^{-1}(\Omega\setminus\Lambda)$ *admits the structure of a Riemann surface on which every function in* A *is analytic.*

PROOF. Fix a function $g \in A$. By hypothesis, if $\lambda \in E$, $\#f^{-1}(\lambda) \leq n$; so $\#g(f^{-1}(\lambda)) \leq n$ and hence $d_{n+1}(g(f^{-1}(\lambda))) = 0$. We define $\psi(\lambda) = \log d_{n+1}(g(f^{-1}(\lambda)))$ for $\lambda \in \Omega$. Then $\psi(\lambda) = -\infty$ on E. Also by Theorem 11.6, ψ is subharmonic on Ω. By the Appendix, since $c(E) > 0$, this implies that ψ is identically equal to $-\infty$. Hence $d_{n+1}(g(f^{-1}(\lambda))) = 0$ for all $\lambda \in \Omega$.

Fix $\lambda_0 \in \Omega$. Suppose that $\#f^{-1}(\lambda_0) \geq n+1$. Then, because A separates the points of X, we may choose $g \in A$ such that the set $g(f^{-1}(\lambda_0))$ contains at least $n+1$ points. Hence $d_{n+1}(g(f^{-1}(\lambda_0))) \neq 0$. This is a contradiction; so no such λ_0 exists. Thus $\#f^{-1}(\lambda) \leq n$ for every $\lambda \in \Omega$. Assertion (i) is proved. □

Define $\Omega_n = \{\lambda : \#f^{-1}(\lambda) = n\}$. We clearly may assume that Ω_n is nonempty. Fix $p \in f^{-1}(\Omega_n)$. We shall construct a neighborhood of p in X such that f maps this neighborhood one-one onto a disk in Ω, centered at $\lambda_0 = f(p)$.

By hypothesis, $f^{-1}(\lambda_0) = \{p_1, p_2, \ldots, p_n\}$. Without loss of generality, $p = p_1$. We choose disjoint compact neighborhoods $\mathcal{U}_j$ of p_j, $1 \leq j \leq n$, and choose a closed disk $\Delta = \{\lambda : |\lambda - \lambda_0| \leq r_0\}$ with $f^{-1}(\Delta) \subseteq \cup_{j=1}^n \mathcal{U}_j$. For each j, we put $X_j = f^{-1}(\Delta) \cap \mathcal{U}_j$. Then $f^{-1}(\Delta) = \cup_{j=1}^n X_j$. By shrinking Δ, we may assume that there exists $h \in A$, such that the sets $h(X_j)$, $1 \leq j \leq n$, lie in disjoint closed disks in $\mathbb{C}$.

Fix $g \in A$. Define $\tilde{g}$ on $f^{-1}(\Delta)$ by

$$\tilde{g} = \begin{cases} g & \text{on } X_1 \\ 0 & \text{on } \cup_{j\neq 1} X_j. \end{cases}$$

Claim 1. $\tilde{g}$ is the uniform limit on $f^{-1}(\Delta)$ of a sequence of functions in A.

PROOF. Since the sets $h(X_j)$, $1 \leq j \leq n$, lie in disjoint closed disks, we can choose a sequence of polynomials $\{P_n\}$ such that $P_n \to 1$ uniformly on $h(X_1)$ and $P_n \to 0$ uniformly on $h(X_j)$, for each $j \neq 1$. Hence the sequence $(P_n \circ h)g$ tends to $\tilde{g}$ uniformly on $\cup_{j=1}^n X_j = f^{-1}(\Delta)$. Since $(P_n \circ h)g \in A$ for each n, Claim 1 is proved. □

Claim 2. Fix a closed disk T, with center λ_1, contained in $\operatorname{int}\Delta$. Fix $x_1 \in f^{-1}(\lambda_1) \cap X_1$. Then

$$|g(x_1)| \leq \max_{f^{-1}(\partial T)\cap X_1} |g|.$$

PROOF. In view of Claim 1, there exists a sequence $\{g_k\}$ in A tending uniformly to $\tilde{g}$ on each X_j, $1 \leq j \leq n$. For each k, $|g_k(x_1)| \leq \max_{f^{-1}(\partial T)} |g_k|$. Letting $k \to \infty$, we get $|\tilde{g}(x_1)| \leq \max_{f^{-1}(\partial T)} |\tilde{g}|$, and so $|g(x_1)| \leq \max_{f^{-1}(\partial T)\cap X_1} |g|$, as claimed. □

Claim 3. Fix r, $0 < r \leq r_0$. Let T be the disk $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| \leq r\} \subseteq \operatorname{int}\Delta$. Then

$$f(f^{-1}(\partial T) \cap X_1) = \partial T.$$

PROOF. Suppose not. Then $f(f^{-1}(\partial T) \cap X_1)$ is a proper closed subset γ_0 of ∂T. We may choose a polynomial Q with $|Q(\lambda_0)| > \max_{\gamma_0} |Q|$. Now $p_1 \in f^{-1}(\lambda_0) \cap X_1$. By Claim 2,

$$|(Q \circ f)(x_1)| \leq \max_{f^{-1}(\partial T) \cap X_1} |Q \circ f|.$$

Also, $|(Q \circ f)(x_1)| = |Q(\lambda_0)| > \max_{f^{-1}(\partial T) \cap X_1} |Q \circ f|$. This is a contradiction; hence Claim 3 holds. □

It is clear that in the last three claims we may replace X_1 by any X_j. We now put $X_j^0 = X_j \cap f^{-1}(\operatorname{int}(\Delta))$, $1 \leq j \leq n$. We denote by f_j the restriction of f to X_j^0 and by A_j the restriction of the algebra A to X_j^0. Claim 3 yields that $f(X_j^0) \supseteq \operatorname{int}\Delta$, for all j. In view of the last three claims, $(A_j, X_j^0, \operatorname{int}\Delta, f_j)$ is a maximum modulus algebra over $\operatorname{int}\Delta$. Since $\# f^{-1}(\lambda) \leq n$ for each $\lambda \in \Omega$, it follows that each X_j^0 is mapped one-one by f_j to $\operatorname{int}\Delta_j$. By Corollary 11.3 of Theorem 11.2, we obtain the following.

Claim 4. If $g \in A$, $1 \leq j \leq n$, there exists G_j analytic on $\operatorname{int}\Delta$ and continuous on Δ such that

$$g = G_j \circ f \qquad \text{on } f_j^{-1}(\Delta).$$

Now we have shown that f is a local homeomorphism from $f^{-1}(\Omega_n)$ to Ω_n. This means that $f^{-1}(\Omega_n)$ is a Riemann surface defined by using f as a local coordinate at each point—the "transition functions" between the local patches are just the identity functions in all cases. Moreover, by Claim 4, the functions in A are analytic with respect to this Riemann surface structure. Finally, by definition of Ω_n, f is an n-to-one map of $f^{-1}(\Omega_n)$ onto Ω.

We shall show that $\Omega \setminus \Omega_n$ is a discrete subset of Ω. Fix $\lambda_0 \in \Omega_n$ and choose $g \in A$ such that g separates the points of $f^{-1}(\lambda_0)$. For $\lambda \in \Omega$, let $p_1, p_2, \ldots, p_n$ denote the n points of $f^{-1}(\lambda)$ and put $D(\lambda) = \prod_{i<j}(g(p_i) - g(p_j))^2$. Note that $D(\lambda)$ is independent of the ordering of the points $\{p_j\}$.

Fix $\lambda_1 \in \Omega_n$. As we saw, there exist a disk Δ centered at λ_1 and neighborhoods $X_1, X_2, \cdots, X_n$ of the points $p_1, p_2, \ldots, p_n$ in $f^{-1}(\lambda_1)$ such that on each X_j, g admits a representation $g = G_j \circ f$, where G_j is analytic on $\operatorname{int}\Delta$. Hence $D(\lambda) = \prod_{i<j}(G_i(\lambda) - G_j(\lambda))^2$ is analytic in λ on Ω_n.

Let b be a point of Ω lying on the boundary of $\Omega \setminus \Omega_n$.

Claim 5. $D(\lambda) \to 0$, as $\lambda \to b$.

PROOF. We need only consider sequences $\{\lambda_k\}$ in Ω_n that converge to b. Without loss of generality, $|g| \leq M$ for some M. Fix $\epsilon > 0$. Since $b \in \partial(\Omega \setminus \Omega_n)$,

$b \in \Omega_s$ for some $s < n$; so $f^{-1}(b) = \{q_1, \cdots, q_s\}$. Choose compact disjoint neighborhoods $\mathcal{U}_j$ of q_j such that $|g(q) - g(q')| \leq \epsilon$ for q and q' in $\mathcal{U}_j$, for each j. For k large, $f^{-1}(\lambda_k) \subseteq \cup_{j=1}^{s} \mathcal{U}_j$. Since $f^{-1}(\lambda_k)$ consists of n points $p_1, \ldots, p_n$, and $n > s$, there exist two points p_α and p_β over λ_k belonging to the same $\mathcal{U}_j$, and so $|g(p_\alpha) - g(p_\beta)| \leq 2\epsilon$. Then

$$|D(\lambda_k)| = \prod_{i<j} |g(p_i) - g(p_j)|^2 \leq \epsilon^2 2^{n(n-1)} M^{n(n-1)-2}.$$

Hence $D(\lambda_k) \to 0$, as $k \to \infty$, so the claim is proved. □

Thus D is continuous on Ω and vanishes on $\partial(\Omega \setminus \Omega_n)$. It follows from Rado's theorem (Theorem 10.9) that D is analytic on Ω. Hence the zero set of D is discrete and so, if $\Lambda = \Omega \setminus \Omega_n$, then Λ is a discrete subset of Ω, as claimed. This proves assertion (ii) of Theorem 11.8.

Exercise 11.5. Let (A, X, Ω, f) be a maximum modulus algebra such that, for some integer n, $\#f^{-1}(\lambda) \leq n$ for all $\lambda \in \Omega$ and such that there exists $\lambda_0 \in \Omega$ with $\#f^{-1}(\lambda_0) = n$. Show that, for all $g \in A$, there exist analytic functions $a_1, a_2, \ldots, a_n$ on Ω such that

$$g^n + a_1(f)g^{n-1} + \cdots + a_n(f) = 0$$

on X.

We now shall consider a special class of maximum modulus algebras, which arise in the study of uniform algebras.

Let $\mathfrak{A}$ be a uniform algebra on a compact space Y and denote by $\mathcal{M}$ the maximal ideal space of $\mathfrak{A}$. Fix a function $f \in \mathfrak{A}$.

The image $f(Y)$ of Y under f is a compact subset of $\mathbb{C}$. We fix an open set $\Omega \subseteq \mathbb{C} \setminus f(Y)$ such that $f^{-1}(\Omega) = \{m \in \mathcal{M} : f(m) \in \Omega\}$ is nonempty. Then $f^{-1}(\Omega)$ is a locally compact Hausdorff space. We write A for the restriction of $\hat{\mathfrak{A}}$ to $f^{-1}(\Omega)$.

Theorem 11.9. *$(A, f^{-1}(\Omega), \Omega, f)$ is a maximum modulus algebra.*

This assertion is an immediate consequence of Rossi's Local Maximum Modulus Principle, Theorem 9.3.

Exercise 11.6. Use Theorem 9.3 to give a proof of Theorem 11.9.

An elementary proof of Theorem 11.9, independent of Theorem 9.3, was found by Slodkowski [Sl2]. We now shall give a version of that proof.

Proof of Theorem 11.9. It is clear that f gives a proper map of $f^{-1}(\Omega)$ to Ω. We also need that f maps $f^{-1}(\Omega)$ onto Ω. Suppose otherwise. Then, since $f^{-1}(\Omega)$ is nonempty, there exists a $\lambda_0 \in \Omega \setminus f(\mathfrak{M})$ such that $\mathrm{dist}(\lambda_0, f(\mathfrak{M})) <$

$\text{dist}(\lambda_0, f(Y))$. Hence there exists $x^0 \in \mathfrak{M}$, with $f(x^0) \in \Omega$ and $|f(x^0) - \lambda_o| = \text{dist}(\lambda_0, f(\mathfrak{M}))$. Then $g \equiv 1/(f - \lambda_0) \in \mathfrak{A}$ and $|g(x^0)| = 1/|f(x^0) - \lambda_0| > \sup_{y\in Y} 1/|f(y) - \lambda_0| = ||g||_Y$, a contradiction. Thus f maps $f^{-1}(\Omega)$ onto Ω.

Now fix a closed disk $\Delta \subseteq \Omega$, with center λ_0. Choose a point $x^0 \in f^{-1}(\lambda_0)$. Let μ_0 be a representing measure for x^0 on Y. For each $\lambda \in \Delta$, we define a functional m_λ on $\mathfrak{A}$ by putting

$$m_\lambda(h) = \int_Y \left(\frac{f - \lambda_0}{f - \lambda}\right) h d\mu_0, \qquad h \in \mathfrak{A}.$$

Then

$$m_{\lambda_0}(h) = h(x^0), \qquad h \in \mathfrak{A}. \tag{29}$$

We next fix $\lambda \in \Delta$ and denote by I_λ the closed ideal in $\mathfrak{A}$, which is the closure of

$$(f - \lambda)\mathfrak{A} = \{h \in \mathfrak{A} : \exists g \in \mathfrak{A} \text{ with } h = (f - \lambda)g\}.$$

We form the quotient algebra $\mathfrak{A}/I_\lambda$ and write $||[h]||_\lambda$ for the quotient norm of the coset $[h]$ in $\mathfrak{A}/I_\lambda$, for $h \in \mathfrak{A}$. Put

$$C = \max_{\lambda\in\Delta, y\in Y} \left|\frac{f(y) - \lambda_0}{f(y) - \lambda}\right|.$$

Then, putting $||h|| = \max_Y |h|$,

$$|m_\lambda(h)| \leq C||h||, \qquad h \in \mathfrak{A}.$$

Also,

$$m_\lambda((f - \lambda)g) = \int_Y (f - \lambda_0) g d\mu_0 = 0, \qquad g \in \mathfrak{A},$$

so $m_\lambda = 0$ on I_λ. Fix $h \in \mathfrak{A}$. For each k in the coset $[h]$, we have

$$|m_\lambda(h)| \leq C||k||;$$

therefore, we get

$$|m_\lambda(h)| \leq C||[h]||_\lambda, \qquad h \in \mathfrak{A}. \tag{30}$$

Finally, the definition of m_λ yields that we have

$$\lambda \mapsto m_\lambda(h) \text{ is analytic on } \Delta \text{ for each } h \in \mathfrak{A}. \tag{31}$$

Now fix $g \in \mathfrak{A}$ with $g(x^0) \neq 0$, and fix $\lambda \in \Delta$. For $n = 1, 2, \ldots,$

$$|m_\lambda(g^n)| \leq C||[g^n]||_\lambda = C||[g]^n||_\lambda.$$

If we apply (31) to $h = g^n$, we may conclude that

$$\lambda \mapsto \log|m_\lambda(g^n)|$$

is subharmonic on Δ. Hence

$$\log|g^n(x^0)| = \log|m_{\lambda_0}(g^n)| \leq \frac{1}{2\pi}\int_{\partial\Delta} \log|m_\lambda(g^n)| d\theta.$$

Therefore, using (30), we get

$$\log |g(x^0)| \leq \frac{1}{2\pi} \int_{\partial\Delta} \frac{1}{n} \log[C||[g^n]||_\lambda] d\theta.$$

It follows that

$$(32) \qquad \log |g(x^0)| \leq \frac{1}{2\pi} \int_{\partial\Delta} \frac{1}{n} \log C d\theta + \frac{1}{2\pi} \int_{\partial\Delta} \log[||[g^n]||_\lambda^{1/n}] d\theta.$$

Now $\mathfrak{A}/I_\lambda$ is a commutative Banach algebra. Each point of $f^{-1}(\lambda)$ induces a homomorphism of $\mathfrak{A}/I_\lambda$ into $\mathbb{C}$. We leave it to the reader to verify that, conversely, each such homomorphism arises in this way. Hence the maximal ideal space of $\mathfrak{A}/I_\lambda$ may be identified with $f^{-1}(\lambda)$. By the spectral radius formula, then,

$$\lim_{n\to\infty} [||[g]^n||_\lambda]^{\frac{1}{n}}$$

exists and equals $\max_{x\in f^{-1}(\lambda)} |g(x)|$. Letting $n \to \infty$ in (32) and using Fatou's Lemma, we get

$$\log |g(x^0)| \leq \frac{1}{2\pi} \int_{\partial\Delta} \log[\max_{f^{-1}(\lambda)} |g|] d\theta.$$

It follows that

$$\log |g(x^0)| \leq \log[\max_{f^{-1}(\partial\Delta)} |g|],$$

and so

$$|g(x^0)| \leq \max_{f^{-1}(\partial\Delta)} |g|.$$

This holds for each g, x^0, and Δ. Therefore, $(A, f^{-1}(\Omega), \Omega, f)$ is a maximum modulus algebra and Theorem 11.9 is proved. □

In the next results we again let $\mathfrak{A}$ be a uniform algebra on the space Y, $\mathcal{M}$ the maximal ideal space of $\mathfrak{A}$, and f an element of $\mathfrak{A}$. As a corollary of Theorem 11.8 and Theorem 11.9, we obtain:

Corollary 11.11. *Let $\mathcal{U}$ be a connected component of $\mathbb{C} \setminus f(Y)$. Assume that, for some integer $n > 0$, $\#f^{-1}(\lambda) \leq n$ for each $\lambda \in \mathcal{U}$. Then there exists a discrete subset Λ of $\mathcal{U}$ such that $f^{-1}(\mathcal{U} \setminus \Lambda)$ admits the structure of a Riemann surface on which every function in $\hat{\mathfrak{A}}$ is analytic.*

Assume $\mathfrak{A}$, Y, $\mathcal{M}$, f as above. Let $\mathcal{U}$ be a connected component of $\mathbb{C} \setminus f(Y)$. Let m denote the arc-length measure.

Theorem 11.12. *Assume that $\partial\mathcal{U}$ contains a smooth arc α as an open subset, and assume that there exists a closed subset E of α such that $m(E) > 0$ and such that $\#f^{-1}(\lambda) \leq n$ for for each $\lambda \in E$. Then $\#f^{-1}(\zeta) \leq n$ for each $\zeta \in \mathcal{U}$, and* *hence the conclusion of Theorem 11.8 holds for the maximum modulus algebra* $(A, f^{-1}(\Omega), \Omega, f)$ *given by Theorem 11.9 with* $\Omega = \mathcal{U}$.

Proof. Recall the k-diameter d_k as in Definition 11.2. Suppose that $f^{-1}(\zeta) > n$ for some $\zeta \in \mathcal{U}$. We construct a region $\mathcal{U}_0$, with a piecewise smooth boundary, such that $\mathcal{U}_0$ contains α as a part of its boundary and with $\zeta \in \mathcal{U}_0 \subseteq \mathcal{U}$.

Claim. Fix $\lambda_0 \in E$. Fix $g \in A$. Put

$$F(\lambda) = \log[d_{n+1}(g(f^{-1}(\lambda))], \qquad \lambda \in \mathcal{U}_0.$$

Then $\limsup_{\lambda \to \lambda_0, \lambda \in \mathcal{U}_0} F(\lambda) = -\infty$.

Proof of Claim. Denote by $q_1(\lambda_0), q_2(\lambda_0), \cdots, q_k(\lambda_0)$ the points of $f^{-1}(\lambda_0)$. By choice of λ_0, $k \leq n$. Next consider the points $g(q_j(\lambda_0))$, $j = 1, \dots, k$ in $\mathbb{C}$. Consider closed disks Δ_j with center $g(q_j(\lambda_0))$, $j = 1, \dots, k$, and radius ϵ, for a small fixed $\epsilon > 0$, where the disks Δ_j are disjoint or coincide, according to whether the points $g(q_j(\lambda_0))$, $j = 1, \dots, k$ are distinct or not. Let $\{\lambda_\nu\}$ be a sequence in $\mathcal{U}_0$ converging to λ_0. Choose a neighborhood $\mathcal{N}$ of $f^{-1}(\lambda_0)$ in $\mathcal{M}$ such that $g(\mathcal{N}) \subseteq \cup_{j=1}^k \Delta_j$.

Choose ν_0 such that, for $\nu \geq \nu_0$, $f^{-1}(\lambda_\nu) \subseteq \mathcal{N}$. Fix $\nu \geq \nu_0$. Then $g(f^{-1}(\lambda_\nu)) \subseteq \cup_{j=1}^k \Delta_j$.

Suppose first that there are $n+1$ distinct points $z_1, \cdots, z_{n+1}$ in $g(f^{-1}(\lambda_\nu))$. Since $k \leq n$, then at least two of them lie in the same Δ_j. Without loss of generality we may assume that z_1, z_2 is such a pair, and so $|z_1 - z_2| \leq 2\epsilon$. Also, for every pair of indices α, β with $1 \leq \alpha, \beta \leq k$,

$$|z_\alpha - z_\beta| \leq 2||g||.$$

Hence

$$\prod_{\alpha<\beta} |z_\alpha - z_\beta| \leq (2\epsilon)(2||g||)^{\frac{(n+1)n}{2}-1}.$$

By definition of d_{n+1}, then

$$d_{n+1}(g(f^{-1}(\lambda_\nu))) \leq ((2\epsilon)(2||g||)^{\frac{(n+1)n}{2}-1})^{\frac{2}{(n+1)n}},$$

and hence

$$F(\lambda_\nu) = \log[d_{n+1}(g(f^{-1}(\lambda_\nu)))] \leq \frac{2}{(n+1)n} \log((2\epsilon)(2||g||)^{\frac{(n+1)n}{2}-1}). \tag{33}$$

On the other hand, if there do not exist $n+1$ distinct points in $g(f^{-1}(\lambda_\nu))$, then $F(\lambda_\nu) = -\infty$. Hence (33) holds for all $\nu \geq \nu_0$. Hence $\limsup_{\nu\to\infty} F(\lambda_\nu)$ has the same bound. This holds for all $\epsilon > 0$, and so $\limsup_{\nu\to\infty} F(\lambda_\nu) = -\infty$. The claim follows. □

We now need the following result on subharmonic functions from the Appendix, A3.

Proposition. *Given $\mathcal{U}_0$, α, E as above, and given a subharmonic function χ defined on $\mathcal{U}_0$, suppose that*

$$\limsup_{\lambda\to\lambda_0,\lambda\in\mathcal{U}_0} \chi(\lambda) = -\infty$$

for each $\lambda_0 \in E$. Then $\chi \equiv -\infty$ in $\mathcal{U}_0$.

By hypothesis, there exist $n+1$ distinct points $p_1, p_2, \cdots, p_{n+1} \in f^{-1}(\zeta)$. We choose $g \in \mathfrak{A}$ with $g(p_1), g(p_2), \cdots, g(p_{n+1})$ distinct points in $\mathbb{C}$. Put $F(\lambda) = \log[d_{n+1}(g(f^{-1}(\lambda)))]$, $\lambda \in \mathcal{U}_0$, as before. By Theorem 11.7, F is subharmonic in $\mathcal{U}_0$. Also, the set $g(f^{-1}(\zeta))$ contains the $n+1$ distinct points $g(p_j)$, $j = 1, 2, \cdots, n+1$. Hence $d_{n+1}(g(f^{-1}(\zeta))) \neq 0$, and so $F(\zeta) > -\infty$.

On the other hand, by the claim and the proposition, $F \equiv -\infty$ in $\mathcal{U}_0$, and so in particular $F(\zeta) = -\infty$. This is a contradiction. Hence $\#f^{-1}(\zeta) \leq n$ for all $\zeta \in \mathcal{U}$, and Theorem 11.12 is proved. $\square$

It will be useful in some applications to derive the old inequality

$$|g(x^0)| \leq \max_{p^{-1}(\partial\Delta)} |g|, \qquad g \in A, \tag{2}$$

for each $x^0 \in p^{-1}(\lambda_0)$ and $g \in A$, which is assumed to hold for every closed disk $\Delta \subseteq \Omega$, from the following weaker assumption:

$$\forall\lambda_0 \in \Omega, \exists\, \epsilon(\lambda_0) > 0 \text{ such that } |g(x^0)| \leq \max_{p^{-1}(\partial\Delta)} |g|, \qquad g \in A \tag{34}$$

for each $x^0 \in p^{-1}(\lambda_0)$, and $g \in A$ for every closed disk Δ with center λ_0 and radius $r < \epsilon(\lambda_0)$.

Proposition 11.13. *Let (A, X, Ω, p) satisfy all of the assumptions for a maximum modulus algebra except for* (2). *Suppose that* (34) *holds. Then* (2) *holds as well, and (A, X, Ω, p) is a maximum modulus algebra on X with projection p.*

Proof. Fix $\lambda_0 \in \Omega$. Choose $\epsilon(\lambda_0)$ as in (34). Let $\Delta = \{\lambda : |\lambda - \lambda_0| \leq r\}$, where $r \leq \epsilon(\lambda_0)$. Fix $g \in A$ and define Z_g as in Definition 11.1. It is clear from the definitions that Z_g, and hence $\log Z_g$, are upper semi-continuous on Ω.

It follows that we can choose a sequence $\{u_k\}$ of continuous functions on Δ such that $u_k \downarrow \log Z_g$ on Δ. For each k, let Q_k be a polynomial in λ such that $|\mathrm{Re}\, Q_k - u_k| \leq 1/k$ on $\partial\Delta$. Then

$$\log Z_g \leq \mathrm{Re}\, Q_k + 1/k \text{ on } \partial\Delta. \tag{35}$$

Fix $x^0 \in p^{-1}(\lambda_0)$. We now apply (34) to g replaced by $ge^{-(Q_k(p)+1/k)}$. Strictly speaking, since $e^{-(Q_k(p)+1/k)}$ need not be in the algebra A, we first apply (34) with the exponential function replaced by the partial sums (polynomials) of its Taylor series and then take a limit. We get

$$|g(x^0)|e^{-(\mathrm{Re}(Q_k(\lambda_0))+1/k)} \leq |ge^{-(Q_k(p)+1/k)}(y)|$$

for some $y \in p^{-1}(\partial\Delta)$. Hence, with $\lambda = p(y)$,

$$|g(x^0)|e^{-(\mathrm{Re}(Q_k(\lambda_0))+1/k)} \leq Z_g(\lambda)e^{-(\mathrm{Re}(Q_k(\lambda))+1/k)} \leq 1,$$

by (35). Hence

$$|g(x^0)| \leq e^{\mathrm{Re}(Q_k(\lambda_0))+1/k}.$$

Since this holds for every $x^0 \in p^{-1}(\lambda_0)$, we have

$$\begin{aligned}\log Z_g(\lambda_0) &\leq \mathrm{Re}\ Q_k(\lambda_0) + 1/k \\ &= \frac{1}{2\pi}\int_{\partial\Delta} \mathrm{Re}(Q_k + 1/k)d\theta \leq \frac{1}{2\pi}\int_{\partial\Delta}(u_k + 2/k)d\theta.\end{aligned}$$

Letting $k \to \infty$ and using the monotone convergence theorem, we arrive at

$$\log Z_g(\lambda_0) \leq \frac{1}{2\pi}\int_{\partial\Delta} \log Z_g d\theta. \tag{36}$$

It follows from (36) (see the Appendix) that $\log Z_g$ is subharmonic on Ω. By the maximum principle for subharmonic functions, this implies (2) for g and gives the proposition. □

12

Hulls of Curves and Arcs

In this chapter we shall study the polynomial hull of a curve in $\mathbb{C}^N$, or, more generally, of a finite union of curves in $\mathbb{C}^N$, by making use of the results on maximum modulus algebras that we gave in the preceding chapter.

Let $\gamma_1, \gamma_2, \ldots, \gamma_p$ be a finite collection of compact smooth curves in $\mathbb{C}^N$ and let γ be their union. We shall write $\hat{\gamma}$ for the polynomial hull $h(\gamma)$.

Theorem 12.1. *If γ is not polynomially convex, then $\hat{\gamma} \setminus \gamma$ is a one-dimensional analytic subvariety of $\mathbb{C}^N \setminus \gamma$.*

We use z_j, $1 \leq j \leq N$, for the complex coordinates in $\mathbb{C}^N$. In order to prove Theorem 12.1 we shall carry out the following steps.

Step 1. Fix a point $x^0 \in \hat{\gamma} \setminus \gamma$. Construct a polynomial f in $z_1, \ldots, z_N$ such that $f(x^0) = 0$ and $0 \notin f(\gamma)$.

Notation. For S a subset of $\mathbb{C}$,

$$f^{-1}(S) = \{(z_1, \ldots, z_N) \in \hat{\gamma} : f(z_1, \ldots, z_N) \in S\}.$$

We say that $f^{-1}(S)$ *lies at most k-sheeted over S* if, for each $\lambda \in S$, $\# f^{-1}(\lambda) \leq k$. We say that $f^{-1}(S)$ *lies finite-sheeted over S* if $f^{-1}(S)$ lies at most k-sheeted over S for some k.

Step 2. Let U and V be components of $\mathbb{C} \setminus f(\gamma)$ that share a common boundary arc α such that, for some integer s, there are exactly s points of γ which are mapped to each point of α. (We will say that *$f|\gamma$ is s to 1 over α*.) Show that if $f^{-1}(U)$ lies at most k-sheeted over U, then $f^{-1}(V)$ lies at most $(k + s)$-sheeted over V.

Step 3. Let U_0 denote the unbounded component of $\mathbb{C} \setminus f(\gamma)$. Show that $f^{-1}(U_0)$ lies at most 0-sheeted over U_0; i.e., if $\lambda \in U_0$, then there exists no point z in $\hat{\gamma}$ with $f(z) = \lambda$.

Step 4. Let U^* denote the component of $\mathbb{C} \setminus f(\gamma)$ that contains 0. Show that there exists a sequence

$$U_1, U_2, \ldots, U_\ell$$

of components of $\mathbb{C} \setminus f(\gamma)$ such that

(1) U_0 and U_1 share a boundary arc α_0 such that $f|\gamma$ is s_0 to 1 over α_0 for some positive integer s_0.

(2) U_j and U_{j+1} share a boundary arc α_j such that $f|\gamma$ is s_j to 1 over α_j for some positive integer s_j, $j = 1, 2, \ldots, \ell - 1$.

(3) $U_\ell = U^*$.

PROOF OF THEOREM 12.1. First we get the sequence

$$U_1, U_2, \ldots, U_\ell = U^*$$

from Step 4. By Step 3, $f^{-1}(U_0)$ lies at most 0-sheeted over U_0. Then, using Step 2 at each of the "edges" $\alpha_0, \alpha_1, \ldots, \alpha_{\ell-1}$ we deduce, after ℓ applications of Step 2, that $f^{-1}(U^*)$ lies finite-sheeted over U^*. By Corollary 11.11, $f^{-1}(U^*)$ is a finite-sheeted analytic cover of U^*. Since $x^0 \in f^{-1}(U^*)$, $f^{-1}(U^*)$ provides a neighborhood of x^0 in $\hat{\gamma}$. Thus $\hat{\gamma}$ is locally an analytic variety at x^0. Since this holds for each $x^0 \in \hat{\gamma} \setminus \gamma$, $\hat{\gamma} \setminus \gamma$ is a one-dimensional analytic subvariety of $\mathbb{C}^N \setminus \gamma$ and we see that Steps 1 through 4 imply Theorem 12.1.

We now proceed to carry out Steps 1 through 4.

We need, for Step 1, to define $R(X)$ for X a compact subset of $\mathbb{C}^N$ as the uniform closure in $C(X)$ of the rational functions $r = p/q$, where p and q are polynomials in $z_1, z_2, \cdots, z_N$ and $q(z) \neq 0$ for $z \in X$. We shall use the observation that, if $z_j(X) \subseteq \mathbb{C}$ has zero planar measure for all j, $1 \leq j \leq N$, then $R(X) = C(X)$. Indeed the assumption implies, by the Hartogs–Rosenthal theorem (Theorem 2.8), that the function $\lambda \mapsto \bar{\lambda}$ can be uniformly approximated on $z_j(X)$ by rational functions of λ with poles off of $z_j(X)$. Composing these rational functions with z_j we see that the function $z \mapsto \bar{z_j}$ belongs to $R(X)$ for each j. Now the Stone–Weierstrass theorem yields $R(X) = C(X)$.

Now suppose that $x^0 \notin \gamma$. Set $X = \{x^0\} \cup \gamma$. Since γ is smooth, $g(X)$ has zero planar measure for all polynomials g on $\mathbb{C}^N$. By the previous paragraph, we have that $R(X) = C(X)$. The function on X that is 1 at x^0 and $= 0$ on γ is continuous on X. Uniformly approximating it by a rational function, we get $r = p/q$ such that $r(x^0) = 1$ and $|r| < 1$ on γ, and $q \neq 0$ on X. Now set $f = p - q$. We have $f(x^0) = 0$ and $f \neq 0$ on γ, since $r \neq 1$ on γ. This completes Step 1.

We lead up to Step 2 with some discussion and a lemma. Let f be a nonconstant function in the disk algebra $A(\Delta)$, and let p be a point in the open unit disk. Put $f(p) = \lambda_0$. The open mapping principle for an analytic functions tells us that each neighborhood of p in Δ is mapped by f onto a neighborhood of λ_0.

The analogous statement for a uniform algebra is not true in general, as we shall see in Example 12.1 further on. However, a violation of the "open mapping principle" has a consequence for the algebra given in the following lemma.

Lemma 12.2. *Let $(A, X, \mathcal{M})$ be a uniform algebra. Fix $p \in \mathcal{M} \setminus X$, $f \in A$, and put $\lambda_0 = f(p)$. Define K, a subset of $\mathbb{C}$, by*

$$K = \{\lambda : |\lambda - \lambda_0| \leq r, \alpha \leq \arg(\lambda - \lambda_0) \leq \beta\},$$

where $-\pi/2 < \alpha < \beta < 3\pi/2$. Assume that for some compact neighborhood $\mathcal{N}$ of p in $\mathcal{M} \setminus X$ we have $f(\mathcal{N}) \subseteq K$. Then p is not a peak-point of the algebra $\overline{A|_{f^{-1}(\lambda_0)}}$ on the space $f^{-1}(\lambda_0)$.

Proof. By the Local Maximum Modulus Principle,

$$|g(p)| \leq \max_{\partial\mathcal{N}} |g|, \quad g \in A.$$

Hence there exists a representing measure μ for p on $\partial\mathcal{N}$, with

$$g(p) = \int_{\partial\mathcal{N}} g d\mu, \quad g \in A.$$

Choose a function ϕ continuous on K, analytic on $\operatorname{int} K$, such that $\phi(\lambda_0) = 1$ and $|\phi(\lambda)| < 1$ for $\lambda \in K \setminus \{\lambda_0\}$. Then ϕ is a uniform limit on K of polynomials in λ. For $n = 1, 2, \ldots$, define the measure $\mu_n = (\phi \circ f)^n \mu$, on $\partial\mathcal{N}$.

Fix $q \in \partial\mathcal{N} \setminus f^{-1}(\lambda_0)$. Then $f(q) \in K \setminus \{\lambda_0\}$ so $|\phi(f(q))| < 1$ and hence $(\phi \circ f)^n(q) \to 0$ as $n \to \infty$. It follows that, if $g \in A$,

$$g(p) = \int_{\partial\mathcal{N}} g d\mu_n = \int_{\partial\mathcal{N}} g(\phi \circ f)^n d\mu \to \int_{\partial\mathcal{N} \cap f^{-1}(\lambda_0)} g d\mu.$$

If $G \in \overline{A|_{f^{-1}(\lambda_0)}}$, there exists a sequence $\{g_\nu\}$ in A such that $g_\nu \to G$ uniformly on $f^{-1}(\lambda_0)$. Hence

$$G(p) = \int_{\partial\mathcal{N} \cap f^{-1}(\lambda_0)} G d\mu.$$

It follows, since $p \in \mathcal{N}$, that p cannot be a peak-point for $\overline{A|_{f^{-1}(\lambda_0)}}$. □

Example 12.1. Let $(A, X, \mathcal{M})$ be the uniform algebra where X is the torus $T^2 = \{(z, w) \in \mathbb{C} : |z| = |w| = 1\}$, A is the bidisk algebra on T^2, and $\mathcal{M}$ is the closed bidisk Δ^2.

Fix $p = (1, 0) \in \mathcal{M} \setminus X$, and let f be the first coordinate function $(z, w) \mapsto z$. Every neighborhood $\mathcal{N}$ of p in $\mathcal{M}$ maps into $\{|z| \leq 1\}$, and so $f(\mathcal{N})$ contains no neighborhood of $f(p) = 1$.

We note that, as predicted in Lemma 12.2, p is not a peak-point of $\overline{A|_{f^{-1}(1)}}$, since $f^{-1}(1)$ is the disk $\{(1, w) : |w| \leq 1\}$.

We now use Lemma 12.2 to prove the following.

Lemma 12.3. *Fix a compact set X in $\mathbb{C}^N$ and let $A = \mathbf{P}(X)$ denote the uniform closure on X of the polynomials in $z_1, z_2, \ldots, z_N$. Fix $f \in A$. Let U be a component of $\mathbb{C} \setminus f(X)$ such that $f^{-1}(U)$ lies k-sheeted over U.*

Let α be a smooth arc on ∂U such that X lies s-sheeted over α for some positive integer s. Then, for almost all $\lambda_0 \in \alpha$, there are at most $k + s$ points in $\hat{X}$ lying over λ_0.

PROOF.

Claim 1. For almost all $\lambda_0 \in \alpha$ there exists an open triangle S such that $\bar{S} \subseteq U \cup \{\lambda_0\}$, with vertex at λ_0, and there exist k single-valued bounded $\mathbb{C}^N$-valued analytic functions on S:

$$\omega^1, \omega^2, \cdots, \omega^k$$

such that

$$f^{-1}(S) = \bigcup_{\nu=1}^{k} \{\omega^\nu(\lambda) : \lambda \in S\} \tag{4}$$

and

$$\lim_{\lambda \in S \to \lambda_0} \omega^\nu(\lambda) \text{ exists.} \tag{5}$$

To verify Claim 1, we first fix $\tau \in U$ such that $f^{-1}(\tau)$ consists of exactly k distinct points $z_1^0, z_2^0, \cdots, z_k^0$. Let g be a polynomial in $\mathbb{C}^N$ that separates the points $z_1^0, z_2^0, \cdots, z_k^0$. By Exercise 11.5, we form the polynomial

$$P(\lambda, Z) = Z^k + a_{k-1}(\lambda)Z^{k-1} + \cdots + a_0(\lambda)$$

such that, for $\lambda \in U$, the roots of $P(\lambda, Z)$ are the values of g on the k (with multiplicity) points of $f^{-1}(\lambda)$. The coefficient functions $a_0(\lambda), a_1(\lambda), \cdots, a_{k-1}(\lambda)$ are bounded analytic functions on U. Let $D(\lambda)$ be the discriminant function of $P(\lambda, Z)$. If $D(\lambda) \neq 0$, then $f^{-1}(\lambda)$ contains k distinct points. D is a bounded analytic function on U and $D(\tau) \neq 0$. Hence it has a nontangential limit that is different from 0 a.e. on α, say, except on a subset Q_1 of α of measure zero.

It follows that, if $\lambda_0 \in \alpha \setminus Q_1$, there exists an open triangle S contained in U with one vertex at λ_0 and such that S approaches λ_0 nontangentially in U and such that $D \neq 0$ in S. For each $\lambda_1 \in S$ there is a neighborhood $\eta(\lambda_1)$ contained in S and k $\mathbb{C}^N$-valued analytic functions $\omega^1, \omega^2, \cdots, \omega^k$ on $\eta(\lambda_1)$ such that, for all $\lambda \in \eta(\lambda_1)$, $f^{-1}(\lambda)$ consists of the k distinct points $\omega^1(\lambda), \omega^2(\lambda), \cdots, \omega^k(\lambda)$. It follows from the Monodromy Theorem, since S is simply connected, that the functions $\omega^1, \omega^2, \cdots, \omega^k$ on a given initial $\eta(\lambda_1)$ can be analytically continued to give single-valued analytic functions on S. By analytic continuation their values at every $\lambda \in S$ are always in $f^{-1}(\lambda)$. Also, their values are always distinct—for if continuations ω^a and ω^b from $\eta(\lambda_1)$ are equal at some point $\lambda_2 \in S$, then $\omega^a = \omega^b$ on a neighborhood of λ_2 and so $\omega^a = \omega^b$ on all of S. Thus (4) follows.

It remains to establish (5). For each coordinate function z_j we form, as was done above for g, the polynomial

$$P_j(\lambda, Z) = Z^k + a_{k-1,j}(\lambda)Z^{k-1} + \cdots + a_{0,j}(\lambda)$$

associated to z_j; i.e., the roots of $P_j(\lambda, Z)$ are the values of z_j at the k points of $f^{-1}(\lambda)$, for each $\lambda \in U$. The finite set $\{a_{t,j}\}$ of bounded analytic functions have nontangential limits everywhere on α except for, say, a set Q_2 of zero measure. We will show that (5) holds for $\lambda_0 \in \alpha \setminus (Q_1 \cup Q_2)$.

Fix $\lambda_0 \in \alpha \setminus (Q_1 \cup Q_2)$ and the associated triangle S and functions $\omega^1, \omega^2, \cdots, \omega^k$. Fix ν, $1 \leq \nu \leq k$. We verify (5) for ω^ν. Write $\omega^\nu = (h_1, h_2, \cdots, h_N)$, where the h_j are bounded complex-valued analytic functions on S. Fix j. It suffices to show that $\lim_{\lambda \to \lambda_0} h_j(\lambda)$ exists. For all $\lambda \in S$, $h_j(\lambda)$ is a root of $P_j(\lambda, Z)$. As $\lambda \in S$ approaches λ_0, the coefficients of $P_j(\lambda, Z)$ approach the corresponding coefficients of $P_j(\lambda_0, Z)$, by the construction of Q_2. It follows that the roots of $P_j(\lambda, Z)$ approach the roots of $P_j(\lambda_0, Z)$ as $\lambda \in S$ approaches λ_0. Hence the set of all limit points of the values of $h_j(\lambda)$, as $\lambda \in S$ approaches λ_0, is a subset of the finite set consisting of the roots of $P_j(\lambda_0, Z)$. This set L of limit points can be written as

$$L = \bigcap_{m=1}^{\infty} \overline{h_j(\{\lambda \in S : |\lambda - \lambda_0| \leq 1/m\})}.$$

Since the sets $h_j(\{\lambda \in S : |\lambda - \lambda_0| \leq 1/m\})$ are connected, it follows that L is connected. Hence L, being finite and connected (and nonempty), is a single point. This means that $\lim_{\lambda \to \lambda_0} h_j(\lambda)$ exists. This gives Claim 1. $\square$

Fix λ_0 as above. Put

$$\omega^\nu(\lambda_0) = \lim_{\lambda \in S \to \lambda_0} \omega^\nu(\lambda), \qquad \nu = 1, 2, \ldots, k$$

for each ν, and put

$$p_\nu^0 = \omega^\nu(\lambda_0).$$

Then $p_1^0, p_2^0, \cdots, p_k^0$ are points in $f^{-1}(\lambda_0)$. Let $q_1, q_2, \cdots, q_s$ be the s points in $X \cap f^{-1}(\lambda_0)$.

We now fix a point $p \in f^{-1}(\lambda_0)$.

Claim 2. If $p \neq p_\nu^0$ for each ν, $1 \leq \nu \leq k$, and $p \neq q_j$ for each j, $1 \leq j \leq s$, then p is not a peak-point of $\overline{A|_{f^{-1}(\lambda_0)}}$.

Proof of Claim 2. We choose a compact neighborhood $\mathcal{N}$ of p in $\hat{X}$ that excludes $p_1^0, p_2^0, \cdots, p_k^0$ and $q_1, q_2, \cdots, q_s$. Suppose, by way of contradiction, that for $m = 1, 2, \ldots$, $f(\mathcal{N})$ meets $(S \setminus \{\lambda_0\}) \cap \{|\lambda - \lambda_0| \leq 1/m\}$. Then there exists $p_m \in \mathcal{N}$ with $f(p_m) \in S \setminus \{\lambda_0\}$ and $|f(p_m) - \lambda_0| \leq 1/m$. Therefore, $p_m \in f^{-1}(S)$; so, putting $z_m = f(p_m)$; we have, by (4), $p_m = \omega^\nu(z_m)$ for some ν, depending on m, $1 \leq \nu \leq k$. By passing to a subsequence, we may assume that ν is fixed. As $m \to \infty$, $p_m \to \omega^\nu(\lambda_0) = p_\nu^0$. Since $\mathcal{N}$ is compact and each $p_m \in \mathcal{N}$, $p_\nu^0 \in \mathcal{N}$. This contradicts the choice of $\mathcal{N}$. So, for a certain m, $f(\mathcal{N})$ fails to meet $(S \setminus \{\lambda_0\}) \cap \{|\lambda - \lambda_0| \leq 1/m\}$. It follows that $f(\mathcal{N} \cap f^{-1}(\{|\lambda - \lambda_0| \leq 1/m\}) \subseteq \{|\lambda - \lambda_0| \leq 1/m\} \setminus S$. By Lemma 12.2 applied to the neighborhood

$\mathcal{N} \cap f^{-1}(\{|\lambda - \lambda_0| \leq 1/m\})$, then, p is not a peak-point of $\overline{A|_{f^{-1}(\lambda_0))}}$. Claim 2 is proved.

Claim 2 yields that the set of peak-points of $\overline{A|_{f^{-1}(\lambda_0)}}$ is contained in the finite set

$$p_1^0, p_2^0, \cdots, p_k^0, q_1, q_2, \ldots, q_s.$$

Hence $f^{-1}(\lambda_0)$ is a subset of this set. This gives Lemma 12.3. □

We now can complete Step 2. By Lemma 12.3, almost all points $\lambda \in \alpha$ have the property that $f^{-1}(\lambda) \cap \hat{\gamma}$ contains at most $k + s$ points. By the regularity of arclength measure, therefore, there is a compact subset E of α of positive measure such that $f^{-1}(\lambda) \cap \hat{\gamma}$ contains at most $k + s$ points for all $\lambda \in E$. Then Theorem 11.12 implies that $f^{-1}(V)$ lies at most $(k + s)$-sheeted over V. This gives Step 2.

For Step 3 we recall from the first paragraph of the proof of Theorem 11.9 that either $f(\hat{\gamma})$ contains U_0 or $f(\hat{\gamma})$ is disjoint from U_0—we use the fact that, by Exercise 7.4, $\hat{\gamma}$ is the maximal ideal space of $\mathbf{P}(\gamma)$. Since $|f(z)| \leq ||f||_\gamma < \infty$ for all $z \in \hat{\gamma}$, $f(\hat{\gamma})$ does not contain U_0. Hence $f(\hat{\gamma})$ is disjoint from U_0. This gives Step 3.

Finally we carry out Step 4. We can parametrize γ by a finite set of $\mathcal{C}^1$ maps $\phi_j : J_j \equiv [a_j, b_j] \to \mathbb{C}^N$, $1 \leq j \leq n$. Since $0 \notin f(\gamma)$, we have $f \circ \phi_j \neq 0$ and so we have well-defined maps $\psi_j \equiv f \circ \phi_j / |f \circ \phi_j| : J_j \to \Gamma$, where Γ is the unit circle. Let C_j be the set of critical values of ψ_j. This includes $\psi_j(\partial J_j)$. By Sard's theorem (see the Appendix), C_j is a compact subset of Γ of (linear) measure zero. Let C be the union of the C_j; C is also a compact subset of Γ of (linear) measure zero. Hence there exist $-\pi/2 < \alpha < \beta < \pi/2$ such that $\gamma_0 = \{e^{it} : \alpha \leq t \leq \beta\}$ is disjoint from C. By the chain rule, ψ_j is regular on $J_j \setminus \psi_j^{-1}(C)$.

Consider the set $\psi_j^{-1}(\gamma_0) \subseteq J_j$. This set is a union of a finite number of closed intervals $\{\gamma_{j,k}\}$ such that $f \circ \phi_j$ maps each such interval $\gamma_{j,k}$ homeomorphically onto an arc $\sigma_{j,k} \subseteq \mathbb{C}$, where each $\sigma_{j,k}$ is contained in the "wedge" $V = \{\zeta : \alpha \leq \arg \zeta \leq \beta\}$; each $\sigma_{j,k}$ joins the ray $\{\arg \zeta = \alpha\}$ to the ray $\{\arg \zeta = \beta\}$ and meets each ray $\{\arg \zeta = t\}$, $\alpha \leq t \leq \beta$, exactly once. There are, say, K_j such intervals, for each j, $1 \leq j \leq n$.

For $\alpha < t < \beta$, let $m(t)$ be the number of points in the set

$$\{\arg \zeta = t\} \cap \bigcup_{j,k} \sigma_{j,k}.$$

Then $m(t) \leq K_1 + K_2 + \cdots + K_n$. Choose a t_0 such that $m(t_0) = \max_{\alpha<t<\beta} m(t)$. We put $\ell = m(t_0)$. By the continuity of the maps $f \circ \phi_j$, there exists $\epsilon > 0$ such that $m(t) \equiv \ell$ for $t_0 - \epsilon \leq t \leq t_0 + \epsilon$ and there exist ℓ distinct and disjoint arcs σ_s, $1 \leq s \leq \ell$, among the arcs $\{\sigma_{j,k}\} \cap W$, where W is the wedge $\{\zeta : t_0 - \epsilon \leq \arg \zeta \leq t_0 + \epsilon\} \cup \{0\}$. That is, each of the $K_1 + K_2 + \cdots + K_n$ arcs $\sigma_{j,k} \cap W$ is equal to one of the ℓ arcs σ_s. Then the set

$$E = W \setminus \bigcup_{1 \leq s \leq \ell} \sigma_s$$

consists of $\ell + 1$ components $W_0, W_1, \dots, W_\ell$, where W_0 is unbounded and W_ℓ contains 0. Since $f(\gamma) \cap W = E$, each W_s is contained in a component U_s of $\mathbb{C} \setminus f(\gamma)$ for $1 \le s \le \ell$. Clearly, (3) holds since $0 \in W_\ell \subseteq U_\ell$. Also, (1) and (2) hold, by the construction of the $\gamma_{j,k}$ and $\sigma_{j,k}$, since the arcs $\{\alpha_s\}$ of (1) and (2) are just the arcs $\{\sigma_s\}$. This gives Step 4 and concludes the proof of Theorem 12.1.

In the case that γ is a smooth *arc*, we can deduce the following from Theorem 12.1.

Theorem 12.4. *Let γ be a smooth arc in $\mathbb{C}^N$. Then γ is polynomially convex and* $\mathbf{P}(\gamma) = C(\gamma)$.

PROOF. First we show that γ is polynomially convex. Arguing by contradiction, we suppose that there exists $x^0 \in \hat{\gamma} \setminus \gamma$. By Step 1 in the proof of Theorem 12.1, there exists a polynomial f such that $f(x^0) = 0$ and $f \neq 0$ on γ. Since γ is an arc, there exists an open simply connected neighborhood U of γ in $\mathbb{C}^N$ such that f has an analytic logarithm U. That is, there exists an analytic function h defined on U, such that $e^h = f$ on U. By Theorem 12.1, $A = \hat{\gamma} \setminus \gamma$ is a one-dimensional analytic subset of $\mathbb{C}^N \setminus \gamma$.

Choose U_0 open in $\mathbb{C}^N$ such that $\gamma \subseteq U_0 \subseteq \overline{U_0} \subseteq U$. Then $Q = A \setminus U_0$ is a compact subset of A such that the boundary of Q in A lies in U, so that f has a logarithm on the boundary of Q. It follows by the argument principle that f has no zeros on Q. This is a contradiction, since $f(x^0) = 0$ and $x^0 \in Q$. Thus γ is polynomially convex.

Before proceding we shall give a few more details about this application of the argument principle. Except for singular points, A is a Riemann surface. This means that we can triangulate A using "triangles" $\{T\}$ (2-simplices with smooth edges, oriented by the analytic structure) such that the interior of each of these triangles contain only regular points and $f \neq 0$ on ∂T for all T— in particular, x^0 is an interior point of one of these triangles, say T_0. By the classical argument principle in a triangle,

$$\frac{1}{2\pi i} \int_{\partial T} \frac{df}{f} \ge 0$$

for all T, and

$$\frac{1}{2\pi i} \int_{\partial T_0} \frac{df}{f} \ge 1.$$

Let Σ be the union of all the triangles that meet Q. Since

$$\partial \Sigma = \sum \{\partial T : T \text{ meets } Q\},$$

we get

$$\frac{1}{2\pi i} \int_{\partial \Sigma} \frac{df}{f} \ge 1.$$

On the other hand, since $Q \subseteq \Sigma$, after cancellation of common boundaries along the triangles comprising Σ, we have $\partial\Sigma \subseteq U$ and so, since $f = e^h$ on U,

$$\frac{1}{2\pi i}\int_{\partial\Sigma}\frac{df}{f} = \frac{1}{2\pi i}\int_{\partial\Sigma} dh = 0.$$

This is the desired contradiction arising from the argument principle.

We have that γ is polynomially convex. Hence γ is the maximal ideal space of $\mathbf{P}(\gamma)$. This implies that if g is analytic on a neighborhood of $z_j(\gamma) \subseteq \mathbb{C}$, then $g \circ z_j \in \mathbf{P}(\gamma)$. Consequently, the argument used in Step 1 of the proof of Theorem 12.1, to show that $R(X) = C(X)$, shows in the present case that $\mathbf{P}(\gamma) = C(\gamma)$. $\square$

For certain applications it is sometimes useful to have a statement that is more general than Theorem 12.1. The following result reduces to Theorem 12.1 when $K = \emptyset$.

Theorem 12.5. *Let γ be a finite union of smooth compact curves in $\mathbb{C}^N$ (as in Theorem 12.1) and let K be a compact polynomially convex set in $\mathbb{C}^N$. Then $\widehat{K \cup \gamma} \setminus (K \cup \gamma)$ is a (possibly empty) one-dimensional analytic subvariety of $\mathbb{C}^N \setminus (K \cup \gamma)$.*

Sketch of proof. Suppose that there exists $x^0 \in \widehat{K \cup \gamma} \setminus (K \cup \gamma)$. We must show that $\widehat{K \cup \gamma}$ is a one-dimensional analytic set near x^0. The first step is to produce a polynomial f in $z_1, z_2, \cdots, z_N$ such that $f(x^0) = 0$, $0 \notin f(\gamma)$, and $\mathrm{Re}(f) < 0$ on K. This uses the polynomial convexity of K—we omit the details. Now the proof of Theorem 12.1 carries over to give Theorem 12.5. In particular, the construction in Step 4 of the proof of Theorem 12.1 works here because $f(K)$ lies in the left half-plane and the wedge V constructed in Step 4 lies in the right-half plane, and so $f^{-1}(V) \cap (\gamma \cup K) = f^{-1}(V) \cap \gamma$. $\square$

13

Integral Kernels

13.1 Introduction

Let D be a smoothly bounded domain in $\mathbb{C}^n$. By a *kernel* $K(\zeta, z)$ for D we mean a differential form

$$(1) \quad K(\zeta, z) = \sum_{j=1}^{n} a_j(\zeta, z) d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n$$

whose coefficient functions a_j are defined and smooth for $\zeta, z \in \bar{D}$ with $\zeta \neq z$. Here $\widehat{xxx}$ means omit xxx.

We are aiming for a formula

$$(2) \qquad c_0 f(z) = \int_{\partial D} f(\zeta) K(\zeta, z), \quad z \in D,$$

which is valid for every $f \in A(D)$, with c_0 a constant, where $A(D)$ denotes the algebra of functions that are continuous on $\bar{D}$ and holomorphic on D. In the case $n = 1$, we have the celebrated formula

$$(3) \qquad f(z) = \frac{1}{2\pi i} \int_{\partial D} f(\zeta) \frac{d\zeta}{\zeta - z}.$$

Here the kernel $C(\zeta, z) = d\zeta/(\zeta - z)$ is the *Cauchy kernel*. Formula (1) gives a form of type $(n, n-1)$ in ζ, which allows us to integrate expressions $f(\zeta)K(\zeta, z)$ over ∂D.

In all that follows, the differential operators $d, \partial, \bar{\partial}$ will be understood to be taken with respect to the ζ-variables.

We list three properties enjoyed by the Cauchy kernel $C(\zeta, z)$:

(i) $dC = 0$ on $\mathbb{C} \setminus \{z\}$.

(ii) $\int_{|\zeta - z| = \epsilon} C(\zeta, z) = 2\pi i$, $\forall z \in \mathbb{C}$, $\epsilon > 0$.

(iii) For every continuous function g on $\mathbb{C}$,

$$\left| \int_{|\zeta - z| = \epsilon} g(z) C(\zeta, z) \right| \leq 2\pi \max_{|\zeta - z| = \epsilon} |g(\zeta)|,$$

valid for all $z \in \mathbb{C}$, $\epsilon > 0$.

EXERCISE 13.1. Verify properties (i), (ii), and (iii).

We now turn to the case $n > 1$, and consider a smoothly bounded domain D in $\mathbb{C}^n$. We fix a kernel $K(\zeta, z)$ given by (1) and we impose on K the following three conditions, analogous to (i), (ii), (iii):

(4) $dK = 0$ on $D \setminus \{z\}$.

(5) $\int_{|\zeta - z| = \epsilon} K = c_0$, where c_0 is a constant independent of z and ϵ. (Here, $z \in D$ and $\{|\zeta - z| = \epsilon\}$ is the sphere in $\mathbb{C}^n$ of center z and radius ϵ.)

(6) There exists a constant M such that, for every continuous function g on D,

$$\left| \int_{|\zeta - z| = \epsilon} g(\zeta) K(\zeta, z) \right| \leq M \max_{|\zeta - z| = \epsilon} |g(\zeta)|, \quad z \in D, \epsilon > 0.$$

Theorem 13.1. *Assume that K is given by (1) and satisfies (4), (5), and (6). Then, for each $f \in A(D) \cap C^1(\bar{D})$, we have*

$$c_0 f(z) = \int_{\partial D} f(\zeta) K(\zeta, z), \quad z \in D. \tag{7}$$

PROOF. Fix $\epsilon > 0$ and put $D_\epsilon = D \setminus \{|\zeta - z| \leq \epsilon\}$. On $\overline{D_\epsilon}$, K is a smooth $(n, n-1)$-form in ζ.

Fix $f \in A(D) \cap C^1(\bar{D})$. We have on D_ϵ,

$$d(f(\zeta)K(\zeta, z)) = df \wedge K + f dK = df \wedge K$$

by (4). Also, $df \wedge K = \bar{\partial} f \wedge K + \partial f \wedge K = 0$, since $\bar{\partial} f$ vanishes because f is analytic, and $\partial f \wedge K = 0$ since ∂f is of type $(1, 0)$ and K is of type $(n, n-1)$. Stokes' Theorem on D_ϵ now gives

$$\int_{\partial D_\epsilon} f K = \int_{D_\epsilon} d(f K) = 0,$$

or

$$\int_{\partial D} f K - \int_{|\zeta - z| = \epsilon} f K = 0,$$

where $\{|\zeta - z| = \epsilon\}$ is taken with the positive orientation. Thus

$$\begin{aligned} \int_{\partial D} f K &= \int_{|\zeta - z| = \epsilon} f K = \int_{|\zeta - z| = \epsilon} (f(\zeta) - f(z)) K + f(z) \int_{|\zeta - z| = \epsilon} K \\ &= \int_{|\zeta - z| = \epsilon} (f(\zeta) - f(z)) K + c_0 f(z), \end{aligned} \tag{8}$$

by (5). In view of (6),

$$\left| \int_{|\zeta - z| = \epsilon} (f(\zeta) - f(z))K \right| \leq M \max_{|\zeta - z| = \epsilon} |f(\zeta) - f(z)|.$$

Since f is continuous at z,

$$\lim_{\epsilon \to 0} \Big[\int_{|\zeta - z| = \epsilon} (f(\zeta) - f(z))K \Big] = 0.$$

It follows from (8), letting $\epsilon \to 0$, that

$$\int_{\partial D} fK = c_0 f(z).$$

Thus (7) is proved. □

13.2 The Bochner–Martinelli Integral

Fix $n > 1$. How shall we obtain a kernel K satisfying (4), (5), and (6)? In the 1940s, Martinelli and then Bochner (independently) constructed such a kernel. We denote it by K_{MB}. It is defined by

(9) $$K_{MB}(\zeta, z) = \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n.$$

For fixed $z \in \mathbb{C}^n$, the coefficients of K_{MB} are smooth on $\mathbb{C}^n \setminus \{z\}$.

Lemma 13.2. *K_{MB} satisfies (4), (5), and (6) on each smoothly bounded domain $D \subseteq \mathbb{C}^n$. The constant c_0 in (5) equals $(2\pi i)^n/(n-1)!$.*

Proof. For each j, $1 \leq j \leq n$, we put

(10) $$\omega_j = d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_j} \wedge d\zeta_j \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n.$$

Then

$$K_{MB} = \sum_{j=1}^{n} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^{2n}} \omega_j.$$

Exercise 13.2. For each j, we have

$$d\bar{\zeta}_j \wedge \omega_j = \wedge_{j=1}^{n} d\bar{\zeta}_j \wedge d\zeta_j.$$

We put $\beta = \wedge_{j=1}^{n} d\bar{\zeta}_j \wedge d\zeta_j$, and so we have

(11) $$d\bar{\zeta}_j \wedge \omega_j = \beta, \quad j = 1, 2, \cdots, n.$$

We proceed to calculate dK_{MB}:

$$\begin{aligned} dK_{MB} &= \sum_{j=1}^{n} d\Big[\frac{\bar{\zeta}_j - \bar{z}_j}{(|\zeta - z|^2)^n}\Big] \wedge \omega_j \\ &= \sum_{j=1}^{n} \Big[\frac{|\zeta - z|^{2n} d\bar{\zeta}_j - (\bar{\zeta}_j - \bar{z}_j)n(|\zeta - z|^2)^{n-1} d(|\zeta - z|^2)}{|\zeta - z|^{4n}}\Big] \wedge \omega_j. \end{aligned}$$

Now,

$$d(|\zeta - z|^2) = d\Big[\sum_{k=1}^{n}(\zeta_k - z_k)(\bar{\zeta}_k - \bar{z}_k)\Big] = \sum_{k=1}^{n}\big[(\zeta_k - z_k)d\bar{\zeta}_k + (\bar{\zeta}_k - \bar{z}_k)d\zeta_k\big].$$

Since ω_j contains each $d\zeta_k$ as a factor, $d\zeta_k \wedge \omega_j = 0$ for all k. Similarly, $d\bar{\zeta}_k \wedge \omega_j = 0$ unless $j = k$. Finally, $d\bar{\zeta}_j \wedge \omega_j = \beta$ by (11). Hence

$$\begin{aligned} dK_{MB} &= \frac{1}{|\zeta - z|^{4n}} \sum_{j=1}^{n} \\ &\quad \times \big[|\zeta - z|^{2n} d\bar{\zeta}_j \wedge \omega_j - |\zeta_j - z_j|^2 n(|\zeta - z|)^{2n-2}) d\bar{\zeta}_j \wedge \omega_j\big] \\ &= \frac{1}{|\zeta - z|^{4n}} \big[n|\zeta - z|^{2n}\beta - n|\zeta - z|^{2n}\beta\big] = 0. \end{aligned}$$

Thus K_{MB} satisfies (4).

Next, with $z \in \mathbb{C}^n$, $\epsilon > 0$, we have

$$\int_{|\zeta - z| = \epsilon} K_{MB} = \sum_{j=1}^{n} \int_{|\zeta - z| = \epsilon} \frac{1}{\epsilon^{2n}} (\bar{\zeta}_j - \bar{z}_j)\omega_j.$$

Writing, as earlier,

$$\beta = d\bar{\zeta}_j \wedge \omega_j = \wedge_{j=1}^{n} d\bar{\zeta}_j \wedge d\zeta_j,$$

we have by Stokes' Theorem, for each j,

$$\int_{|\zeta - z| = \epsilon} (\bar{\zeta}_j - \bar{z}_j)\omega_j = \int_{|\zeta - z| \le \epsilon} d[(\bar{\zeta}_j - \bar{z}_j)\omega_j] = \int_{|\zeta - z| \le \epsilon} d\bar{\zeta}_j \wedge \omega_j = \int_{|\zeta - z| \le \epsilon} \beta,$$

and so

$$\int_{|\zeta - z| = \epsilon} K_{MB} = \frac{n}{\epsilon^{2n}} \int_{|\zeta - z| \le \epsilon} \beta.$$

We write $\zeta_j = \xi_j + i\eta_j$, $j = 1, 2, \ldots, n$, with ξ_j, η_j real. Then $\beta = \wedge_{j=1}^{n}(d\xi_j - i d\eta_j) \wedge (d\xi_j + i d\eta_j) = (2i)^n d\xi_1 \wedge d\eta_1 \wedge \cdots \wedge d\xi_n \wedge d\eta_n$. Let dx denote Lebesgue measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Then, as a measure on $\mathbb{R}^{2n}$, $\beta = (2i)^n dx$. So

$$\int_{|\zeta - z| = \epsilon} K_{MB} = \frac{n}{\epsilon^{2n}} (2i)^n \int_{|\zeta - z| \le \epsilon} dx = \frac{n}{\epsilon^{2n}} (2i)^n \epsilon^{2n} \mathrm{vol}(B^{2n}),$$

where B^{2n} is the unit ball in $\mathbb{R}^{2n}$. Thus

$$\int_{|\zeta - z| = \epsilon} K_{MB} = n(2i)^n \frac{\pi^n}{n!} = \frac{(2\pi i)^n}{(n-1)!}.$$

Thus (5) holds with $c_0 = \frac{(2\pi i)^n}{(n-1)!}$.

Finally, let g be a continuous function on $\mathbb{C}^n$. Fix $z \in \mathbb{C}^n$, $\epsilon > 0$:

$$\int_{|\zeta - z| = \epsilon} g K_{MB} = \sum_{j=1}^{n} \frac{1}{\epsilon^{2n}} \int_{|\zeta - z| = \epsilon} g(\zeta)(\bar{\zeta}_j - \bar{z}_j)\omega_j.$$

We make the change of variable: $\zeta = z + \epsilon b$, where $b = (b_1, b_2, \cdots, b_n)$ is in $\mathbb{C}^n$. After this change of variable, $\{\zeta : |\zeta - z| = \epsilon\} = \{b : \sum_{j=1}^n |b_j|^2 = 1\}$, and we denote the right-hand side by S. Then, for all j,

$$\begin{aligned}
&\int_{|\zeta - z| = \epsilon} g(\zeta)(\bar{\zeta}_j - \bar{z}_j)\omega_j \\
&\quad = \int_S g(z + \epsilon b)(\epsilon \bar{b}_j)(\epsilon d\bar{b}_1) \wedge (\epsilon d b_1) \wedge \cdots \wedge \widehat{(\epsilon d\bar{b}_j)} \\
&\quad \wedge (\epsilon d b_j) \wedge \cdots \wedge (\epsilon d\bar{b}_n) \wedge (\epsilon d b_n) \\
&\quad = \epsilon^{2n} \int_S g(z + \epsilon b)\sigma_j(b),
\end{aligned}$$

where $\sigma_j(b) = \bar{b}_j d\bar{b}_1 \wedge db_1 \wedge \cdots \wedge \widehat{d\bar{b}_j} \wedge db_j \wedge \cdots \wedge d\bar{b}_n \wedge db_n$. Here, σ_j is a $(2n-1)$-form in b, independent of z and ϵ. It follows that, for some constant k,

$$\left| \int_S h(\theta)\sigma_j \right| \leq k \max_S |h|,$$

for every continuous function h on S. Thus, for each j,

$$\left| \int_S g(z + \epsilon b)\sigma_j(b) \right| \leq k \max_{|\zeta - z| = \epsilon} |g|,$$

and hence

$$\left| \int_{|\zeta - z| = \epsilon} g K_{MB} \right| \leq \sum_{j=1}^{n} \frac{1}{\epsilon^{2n}} \epsilon^{2n} \left| \int_S g(z + \epsilon b)\sigma_j(b) \right| \leq nk \max_{|\zeta - z| = \epsilon} |g|.$$

So K_{MB} satisfies (6), with $M = nk$. We are done. $\square$

In view of Theorem 13.1, the lemma gives

Theorem 13.3. *For each smoothly bounded domain $D \subseteq \mathbb{C}^n$,*

$$\frac{(2\pi i)^n}{(n-1)!} f(z) = \int_{\partial D} f(\zeta) K_{MB}(\zeta, z), \tag{12}$$

whenever $z \in D$ and $f \in A(D) \cap C^1(\bar{D})$.

Remark. It follows immediately from (4) and Stokes' Theorem that $\int_{\partial D} K_{MB}(\zeta, z) = 0$ if $z \in \mathbb{C}^n \setminus \bar{D}$.

13.3 The Cauchy–Fantappie Integral

For various applications, kernels other than the Bochner–Martinelli kernel are desirable. We shall introduce a certain class of kernels.

For a smoothly bounded region $D \subseteq \mathbb{C}^n$, fix functions $w_1(\zeta, z), \dots, w_n(\zeta, z)$, defined and smooth on $\bar{D} \times \bar{D} \setminus \{\zeta = z\}$. Assume that

$$\sum_{j=1}^{n} w_j(\zeta, z)(\zeta_j - z_j) = 1, \quad \zeta \in \bar{D}, z \in \bar{D}, \zeta \neq z. \tag{13}$$

We write d, ∂, $\bar{\partial}$ for the differential operator relative to ζ. We define

$$w(\zeta, z) = (w_1(\zeta, z), w_2(\zeta, z), \dots, w_n(\zeta, z)) \quad \zeta \in \bar{D}, z \in \bar{D}, \zeta \neq z.$$

Thus w is a vector-valued map defined on $\bar{D} \times \bar{D} \setminus \{\zeta = z\}$. Given w as above, satisfying (13), we define the corresponding *Cauchy–Fantappie form* $K_w(\zeta, z)$ by

$$K_w(\zeta, z) = \sum_{j=1}^{n} (-1)^{j-1} w_j dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n. \tag{14}$$

Theorem 13.4 (Leray's Formula). *With D, w as above, we have*

$$f(z) = a_0 \int_{\partial D} f(\zeta) K_w(\zeta, z), \tag{15}$$

for every $f \in A(D) \cap \mathcal{C}^1(\bar{D})$ and $z \in D$, where $a_0 = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$.

We shall deduce formula (15) from the corresponding result (12) for the Bochner–Martinelli kernel. To this end, we now prove a number of lemmas.

Fix a point $(z_1, z_2, \dots, z_n)$ in $\mathbb{C}^n$. We use complex coordinates $Z_1, \dots Z_n$, $W_1, \dots W_n$ in $\mathbb{C}^{2n}$ and define a set Σ_0 in $\mathbb{C}^{2n}$ by

$$\sum_{k=1}^{n} W_k(Z_k - z_k) = 1. \tag{16}$$

If $(Z_1, \dots Z_n, W_1, \dots, W_n)$ lies on Σ_0, $Z_k \neq z_k$ for some k, so we can locally represent Σ_0 by the equation

$$W_k = (1 - \sum_{j \neq k} W_j(Z_j - z_j))(Z_k - z_k)^{-1}.$$

It follows that Σ_0 is a complex manifold of complex dimension $(2n - 1)$.

Lemma 13.5. *Fix positive integers N, k with $k < N$. Let Σ be a k-dimensional complex submanifold of $\mathbb{C}^N$, and let α be a holomorphic k-form on $\mathbb{C}^N$. Denoting* *by* $\alpha|_\Sigma$ *the form on* Σ *obtained by restricting* α *to* Σ*, then,*

$$d\alpha|_\Sigma = 0 \quad \textit{on } \Sigma.$$

PROOF. Let $j : \Sigma \to \mathbb{C}^N$ be the (holomorphic) inclusion map. Then $\alpha|_\Sigma$ is just the "pull back" $j^*(\alpha)$. By a standard property of pull backs, $d(j^*(\alpha)) = j^*(d\alpha)$. Hence $d\alpha|_\Sigma$ is a holomorphic $(k+1)$-form on Σ, being the pull back of the holomorphic $(k+1)$-form $d\alpha$ on $\mathbb{C}^n$. Since there are no nonzero holomorphic $(k+1)$-forms on a complex k-dimensional manifold, we conclude that $d\alpha|_\Sigma = 0$ on Σ. □

We next fix a function $f \in A(D) \cap \mathcal{C}^1(\bar{D})$ and define a form α on $\mathbb{C}^{2n} \cap \{Z \in D\}$ by

$$\text{(17)} \quad \alpha = f(Z) \sum_{k=1}^{n} (-1)^{k-1} W_k dW_1 \wedge \cdots \wedge \widehat{dW_k} \wedge \cdots \wedge dW_n \wedge dZ_1 \wedge \cdots \wedge dZ_n.$$

Then α is a holomorphic $(2n-1)$-form on $\mathbb{C}^{2n} \cap \{Z \in D\}$. We now restrict α to Σ_0, where Σ_0 is given by (16). Lemma 13.5 then gives $d(\alpha|_{\Sigma_0}) = 0$.

We next make a particular choice of functions w_j by putting

$$\tilde{w}_j(\zeta, z) = \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}, \qquad j = 1, 2, \ldots, n.$$

Since

$$\sum_{j=1}^{n} \tilde{w}_j(\zeta, z)(\zeta_j - z_j) = \sum_{j=1}^{n} \frac{|\bar{\zeta}_j - \bar{z}_j|^2}{|\zeta - z|^2} = 1, \qquad \zeta \neq z,$$

condition (13) is satisfied. We form the kernel

$$K_{\tilde{w}}(\zeta, z) =$$

$$\sum_{j=1}^{n} (-1)^{j-1} \frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2} d\left(\frac{\bar{\zeta}_1 - \bar{z}_1}{|\zeta - z|^2}\right)$$

$$\wedge \cdots \wedge \widehat{d\left(\frac{\bar{\zeta}_j - \bar{z}_j}{|\zeta - z|^2}\right)} \wedge \cdots \wedge d\left(\frac{\bar{\zeta}_n - \bar{z}_n}{|\zeta - z|^2}\right) \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

EXERCISE 13.3. Let $P_1, P_2, \ldots, P_n$ and ϕ be functions defined and smooth on an open set $U \subseteq \mathbb{C}^n$, with $\phi \neq 0$ on U. Then we have on U,

$$\sum_{k=1}^{n} (-1)^{k-1} \frac{P_k}{\phi} d\left(\frac{P_1}{\phi}\right) \wedge \cdots \wedge \widehat{d\left(\frac{P_k}{\phi}\right)} \wedge \cdots \wedge d\left(\frac{P_n}{\phi}\right)$$

$$= \frac{1}{\phi^n} \sum_{k=1}^{n} (-1)^{k-1} P_k \, dP_1 \wedge \cdots \wedge \widehat{dP_k} \wedge \cdots \wedge dP_n.$$

Applying the exercise to $P_k = \bar{\zeta}_z - \bar{z}_k$, $k = 1, 2, \ldots, n$, and $\phi = |\zeta - z|^2$, we get

$$K_{\tilde{w}} = \frac{1}{|\zeta - z|^{2n}} \sum_{k=1}^{n} (-1)^{k-1} (\bar{\zeta}_k - \bar{z}_k) d\bar{\zeta}_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_k} \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta,$$

where $d\zeta = d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n$. Recall that the Bochner–Martinelli kernel

$$K_{MB} = \frac{1}{|\zeta - z|^{2n}} \sum_{k=1}^{n} (\bar{\zeta}_k - \bar{z}_k) d\bar{\zeta}_1 \wedge d\zeta_1 \wedge \cdots \wedge \widehat{d\bar{\zeta}_k} \wedge d\zeta_k \wedge \cdots \wedge d\bar{\zeta}_n \wedge d\zeta_n.$$

EXERCISE 13.4. For $q_n = n(n-1)/2$,

$$K_{MB} = (-1)^{q_n} K_{\tilde{w}}$$

We next define a family of maps χ_t, $0 \leq t \leq 1$, from ∂D into $\mathbb{C}^{2n}$, given by

$$\chi_t(\zeta) = (\zeta, t \frac{\bar{\zeta} - \bar{z}}{|\zeta - z|^2} + (1-t) w(\zeta, z)), \quad \zeta \in D.$$

For each ζ,

$$\begin{aligned} &\sum_{k=1}^{n} \Big[t \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^2} + (1-t) w_k(\zeta, z) \Big] (\zeta_k - z_k) \\ &\quad = t \sum_{k=1}^{n} \frac{\bar{\zeta}_k - \bar{z}_k}{|\zeta - z|^2} (\zeta_k - z_k) + (1-t) \sum_{k=1}^{n} w_k(\zeta, z)(\zeta_k - z_k) = 1, \end{aligned}$$

in view of (13) and the corresponding equality for $\tilde{w}$. It follows that the point $\chi_t(\zeta)$ satisfies (16) for each $\zeta \in \partial D$. Thus the cycle $\chi_t(\partial D)$ lies on the manifold Σ_0 defined by (16), for each t, $0 \leq t \leq 1$. It follows that the family of maps χ_t, $0 \leq t \leq 1$, provides a homotopy between the maps

$$\chi_0 : \zeta \longmapsto (\zeta, w(\zeta, z))$$

and

$$\chi_1 : \zeta \longmapsto (\zeta, \tilde{w}(\zeta, z))$$

as maps from ∂D to Σ_0. Hence the cycles $\chi_0(\partial D)$ and $\chi_1(\partial D)$ are homologous in Σ_0. It follows by Stokes' Theorem that we have

$$\int_{\chi_0(\partial D)} \alpha = \int_{\chi_1(\partial D)} \alpha, \tag{18}$$

where α is the form defined by (17), since we have seen that the restriction of α to Σ_0 is a closed form.

From the definition of α and of the maps χ_0 and χ_1, we have that

$$\begin{aligned}\int_{\chi_0(\partial D)} \alpha &\\ &= \int_{\partial D} f(\zeta) \sum_{k=1}^{n} (-1)^{k-1} w_k dw_1 \wedge \cdots \wedge \widehat{dw_k} \wedge \cdots \wedge dw_n \wedge d\zeta \\ &= \int_{\partial D} f(\zeta) K_w(\zeta, z),\end{aligned}$$

and similarly

$$\int_{\chi_1(\partial D)} \alpha = \int_{\partial D} f(\zeta) K_{\tilde{w}}(\zeta, z).$$

By (18), then, the two integrals over ∂D are equal. By Exercise 13.4,

$$K_{\tilde{w}}(\zeta, z) = (-1)^{q_n} K_{MB}(\zeta, z).$$

So

$$\int_{\partial D} f(\zeta) K_w(\zeta, z) = (-1)^{q_n} \int_{\partial D} f(\zeta) K_{MB}(\zeta, z) = (-1)^{q_n} \frac{(2\pi i)^n}{(n-1)!} f(z)$$

by Theorem 13.3. Thus

$$(-1)^{q_n} \frac{(2\pi i)^n}{(n-1)!} f(z) = \int_{\partial D} f(\zeta) K_w(\zeta, z).$$

Theorem 13.4 is proved. □

The last two results will be used in Chapter 14.

Lemma 13.6. *Fix $z \in D$, then $dK_w(\zeta, z) = 0$ for $\zeta \in D \setminus \{z\}$.*

PROOF. Note that (13) yields $d(\sum_{j=1}^{n} (w_j(\zeta, z)(\zeta_j - z_j)) = 0$, or

$$\sum_{j=1}^{n} [\bar{\partial} w_j (\zeta_j - z_j) + \partial w_j (\zeta_j - z_j) + w_j d\zeta_j] = 0. \tag{19}$$

From (14),

$$dK_w = \sum_{j=1}^{n} (-1)^{j-1} dw_j \wedge dw_1 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_n \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

Put

$$\beta_j = \bar{\partial} w_j \wedge \bar{\partial} w_1 \wedge \cdots \wedge \widehat{\bar{\partial} w_j} \wedge \cdots \wedge \bar{\partial} w_n \wedge d\zeta.$$

Then $dK_w = \sum_{j=1}^{n} (-1)^{j-1} \beta_j$. Note that, for all j, $\beta_j = (-1)^{j-1} [\wedge_{k=1}^{n} \bar{\partial} w_k] \wedge \zeta$. Fix $\zeta \neq z$. Without loss of generality, $\zeta_1 \neq z_1$. Equation (19) yields

$$\begin{aligned}(\zeta_1 - z_1)\bar{\partial} w_1 = &- (\zeta_1 - z_1)\partial w_1 - w_1 d\zeta_1 \\ &- \sum_{j \neq 1} [\bar{\partial} w_j (\zeta_j - z_j) + \partial w_j (\zeta_j - z_j) + w_j d\zeta_j].\end{aligned} \tag{20}$$

Let us wedge equation (20) with $(\wedge_{k\neq 1}\bar{\partial} w_k) \wedge d\zeta$. We get on the left $(\zeta_1 - z_1)(\wedge_{k=1}^{n}\bar{\partial} w_k) \wedge d\zeta$ and on the right we get 0, because of repetitions. Hence $(\zeta_1 - z_1)\beta_j = 0$ for all j. It follows that $dK_w = 0$. □

Lemma 13.7. *Fix $z \in D$, and choose $\epsilon > 0$ such that the closed ball $\{|\zeta - z| \le \epsilon\}$ is contained in D. Then*

$$1 = a_0 \int_{\{|\zeta - z| = \epsilon\}} K_w(\zeta, z),$$

where $a_0 = (-1)^{n(n-1)/2}(n-1)!/(2\pi i)^n$.

Proof. Put $D_\epsilon = D \setminus \{|\zeta - z| \le \epsilon\}$. $\int_{\partial D_\epsilon} K_w = \int_{D_\epsilon} dK_w = 0$, by Lemma 13.6. So

$$\int_{\partial D} K_w = \int_{\{|\zeta - z| = \epsilon\}} K_w.$$

By Leray's formula (15),

$$1 = a_0 \int_{\partial D} K_w.$$

Hence

$$1 = a_0 \int_{\{|\zeta - z| = \epsilon\}} K_w,$$

which is the assertion. □

NOTES

The integral representations in this chapter generalize the Cauchy integral formula to smoothly bounded domains in $\mathbb{C}^n$, $n > 1$. Theorem 13.3 was discovered by E. Martinelli in [Mar] and independently by S. Bochner [Boc].

The generalization of the Bochner–Martinelli formula given in Theorem 13.4, which deals with the Cauchy–Fantappie kernels K_w, is due to J. Leray [Ler] (1956) and is based on earlier work by L. Fantappie (1943). For an exposition of the theory and generalizations, see the book of Henkin and Leiterer [HenL], Chapter 1.

14

Perturbations of the Stone–Weierstrass Theorem

Let D be the closed unit disk in $\mathbb{C}$. Given functions f, g on D, we denote by $[f, g]$ the algebra of all functions $P(f, g)$ on D, where P is a polynomial in two variables.

If z is the complex coordinate in $\mathbb{C}$, the Stone–Weierstrass approximation theorem gives that $[z, \bar{z}]$ is dense in $C(D)$. What if we keep the function z, but replace $\bar{z}$ by a function by a function $\bar{z} + R(z)$, where $R(z)$ is a "small" function? Do we then have that $[z, \bar{z} + R]$ is dense in $C(D)$?

If "small" is taken to mean:

$$|R(z)| \leq \epsilon \text{ for all } z \in D$$

with ϵ sufficiently small, the answer is No! Given $\epsilon > 0$, we define

$$R(re^{i\theta}) = \rho(r)e^{-i\theta}, \quad 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi,$$

where $\rho(r) = -r, 0 \leq r \leq \epsilon$ and $\rho(r) = -\epsilon, \epsilon \leq r \leq 1$. Then $|R| \leq \epsilon$ on D, but $\bar{z} + R(z) = 0, |z| \leq \epsilon$. So every function in $[z, \bar{z} + R]$ is analytic on $|z| < \epsilon$, and hence each uniform limit of a sequence of such function on D is analytic there. Hence $[z, \bar{z} + R]$ is not dense in $C(D)$.

However, if "small" is taken in terms of the Lipschitz norm of R, the answer becomes Yes, as will be shown in Theorem 14.3 below. We require two lemmas.

Lemma 14.1. *Given an integer $n \geq 1$, there exists a polynomial P_n so that*

$$\left| P_n(w) - \frac{1}{w + \frac{1}{n}} \right| \leq \frac{1}{n}, \quad w \in S,$$

where S is the closed semidisk

$$\{w \in \mathbb{C} : Re(w) \geq 0 \text{ and } |w| \leq r_0\}, \quad r_0 \text{ fixed.}$$

PROOF. EXERCISE 14.1.

Lemma 14.2. *Let S be as above. There exists a sequence of polynomials $\{P_n\}$ such that*

(1) $$P_n(w) \to \frac{1}{w}, \quad w \in S \setminus \{0\}, \text{ as } n \to \infty, \text{ and}$$

(2) $$|P_n(w)| \le \frac{C}{|w|}, \quad w \in S \setminus \{0\}, n = 1, 2, \ldots,$$

where $C = r_0 + 1$.

PROOF. Let P_n be as in Lemma 14.1. Then

$$\begin{aligned}|wP_n(w)| &\\ &= |w| \left| P_n(w) - \frac{1}{w + \frac{1}{n}} + \frac{1}{w + \frac{1}{n}} \right| \le |w| \left| P_n(w) - \frac{1}{w + \frac{1}{n}} \right| \\ &\quad + \left| \frac{w}{w + \frac{1}{n}} \right| \le \frac{|w|}{n} + 1 \le C, \quad w \in S,\end{aligned}$$

since $|w| < |w + 1/n|$ as $\mathrm{Re}(w) > 0$. Also, for each $w \in S \setminus \{0\}$, $P_n(w) - \frac{1}{w} \to 0$. We are done. □

Theorem 14.3. *Assume that there is a constant $k < 1$ such that*

(3) $$|R(z) - R(z')| \le k|z - z'|, \quad z, z' \in D.$$

Then $[z, \bar{z} + R(z)]$ is dense in $C(D)$.

PROOF. Write $\mathfrak{A} = [z, \bar{z} + R(z)]$. Fix a point $a \in \mathbb{C}$. Let μ be a measure on D with $\mu \perp \mathfrak{A}$. If $|a| > 1$, we have

$$\frac{1}{z - a} = -\sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}},$$

so

(4) $$\int_D \frac{d\mu(z)}{z - a} = -\sum_{n=0}^{\infty} \frac{1}{a^{n+1}} \int_D z^n d\mu(z) = 0.$$

Next assume that $|a| \le 1$ and

(5) $$\int_D \frac{d|\mu|(z)}{|z - a|} < \infty.$$

Note that (5) holds for almost all points a. We shall construct a sequence of elements b_j in $\mathfrak{A}$, $j = 1, 2, \cdots$, such that

(6) $$b_j(z) \to \frac{1}{z - a} \text{ pointwise on } D \setminus \{a\},$$

(7) $$|b_j(z)| \le \frac{C}{|z - a|}, \; z \in D, \; j = 1, 2, \ldots,$$

where C is a constant.

Once this is done we have

$$0 = \int_D b_j(z)d\mu(z) \to \int_D \frac{1}{z-a} d\mu(z),$$

by dominated convergence, in view of (5) and (6). So $\int_D \frac{1}{z-a} d\mu(z) = 0$ a.e. in $\mathbb{C}$. By Lemma 2.7, this implies that $\mu = 0$. It follows that $\mathfrak{A}$ is dense in $C(D)$, as desired.

It remains to construct the functions b_j. We fix $a \in D$. We put

$$h(z) = (z-a)(\bar{z} + R(z) - (\bar{a} + R(a))).$$

Then

$$h(z) = |z-a|^2 + (z-a)(R(z) - R(a)) = |z-a|^2 + B(z). \tag{8}$$

Because of our condition (3),

$$|B(z)| \leq k|z-a|^2 < |z-a|^2, \quad z \in D \setminus \{a\}. \tag{9}$$

Now (8) and (9) give

$$\operatorname{Re} h(z) > 0, \quad z \in D \setminus \{a\}. \tag{10}$$

It follows that $h(D)$ is a compact subset of $\{\operatorname{Re} w \geq 0\}$ and $h(D \setminus \{a\}) \subseteq \{\operatorname{Re} w > 0\}$. We fix a closed semidisk S contained in $\{\operatorname{Re} w \geq 0\}$ that contains $h(D)$.

Next we choose polynomials P_n by Lemma 14.2, satisfying (1) and (2). We then put, for $j = 1, 2, \dots,$

$$b_j(z) = [(\bar{z} + R(z)) - (\bar{a} + R(a))]P_j(h(z)).$$

Since $z \in \mathfrak{A}$ and $\bar{z} + R(z) \in \mathfrak{A}$, also $h \in \mathfrak{A}$, and hence $b_j \in \mathfrak{A}$, for each j.

Fix $z \in D \setminus \{a\}$. As $j \to \infty$,

$$b_j(z) \to [(\bar{z} + R(z)) - (\bar{a} + R(a))] \cdot \frac{1}{h(z)},$$

since $P_j(w) \to \frac{1}{w}$ at $w = h(z)$. So we have

$$b_j(z) \to \frac{1}{z-a}, \quad z \in D \setminus \{a\}.$$

Furthermore, for $z \in D$,

$$|b_j(z)| \leq |[(\bar{z} + R(z)) - (\bar{a} + R(a))]| \cdot \frac{2}{|h(z)|} = \frac{2}{|z-a|}.$$

So (6) and (7) hold, and we are done. □

Exercise 14.2. Show that the hypothesis: $k < 1$, in Theorem 14.3, cannot be weakened.

We now ask: How does Theorem 14.3 generalize when the disk D is replaced by a compact set X in $\mathbb{C}^n$, $n > 1$?

For functions $f_1, f_2, \cdots, f_k$ in $C(X)$ we denote by $[f_1, f_2, \cdots, f_k]$ the algebra of all functions $P(f_1, f_2, \cdots, f_k)$ with P a polynomial in k variables. The Stone–Weierstrass theorem gives that $[z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n]$ is dense in $C(X)$. We now fix functions $R_1, R_2, \cdots, R_n$ and consider the algebra $\mathfrak{A} = [z_1, \cdots, z_n, \bar{z}_1, +R_1 \cdots, \bar{z}_n + R_n]$ of functions on X. When is $\mathfrak{A}$ dense in $C(X)$?

We aim for a sufficient condition on the R_j similar to the hypothesis (3) in Theorem 14.3.

For convenience we assume the existence of a neighborhood N of X such that each R_j is defined in N and lies in $\mathcal{C}^1(N)$. We write $R = (R_1, R_2, \cdots, R_n)$. For $w \in \mathbb{C}^n$, we write $|w| = \sqrt{|w_1|^2 + \cdots + |w_n|^2}$. We shall prove

Theorem 14.4. *Assume that there exists k, $0 \le k < 1$, with*

$$|R(z) - R(z')| \le k|z - z'|, \quad z, z' \in N. \tag{11}$$

Then $\mathfrak{A} = [z_1, \cdots, z_n, \bar{z}_1 + R_1, \cdots, \bar{z}_n + R_n]$ is dense in $C(X)$.

The method of proof consists of replacing the Cauchy kernel $dz/(z - a)$, used in the proof of Theorem 14.3, by a suitably constructed Cauchy–Fantappie kernel $K(\zeta, z)$.

Construction of the Kernel $K(\zeta, z)$

We introduce some notation. For each $\zeta, z \in N$,

$$H_j(\zeta, z) = (\bar{\zeta}_j + R_j(\zeta)) - (\bar{z}_j + R_j(z)), \quad j = 1, 2, \cdots, n. \tag{12}$$

All differential operators d, ∂, $\bar{\partial}$ are with respect to ζ, holding z fixed. Then

$$dH_j = d\bar{\zeta}_j + dR_j(\zeta).$$

As earlier, we put

$$d\zeta = d\zeta_1 \wedge d\zeta_2 \wedge \cdots \wedge d\zeta_n,$$

$$G(\zeta, z) = \sum_{j=1}^{n} H_j(\zeta, z)(\zeta_j - z_j), \tag{13}$$

$$w_j(\zeta, z) = \frac{H_j(\zeta, z)}{G(\zeta, z)} \quad j = 1, 2, \cdots, n. \tag{14}$$

We shall show in (17) below that $G(\zeta, z) \neq 0$ for $\zeta \neq z$. Hence $w_j(\zeta, z)$ is a smooth function on $N \times N \setminus \{\zeta = z\}$. Also, for $\zeta \in N, z \in N, \zeta \neq z$, (14) gives

$$\sum_{j=1}^{n} w_j(\zeta, z)(\zeta_j - z_j) = \sum_{j=1}^{n} \frac{H_j(\zeta, z)}{G(\zeta, z)}(\zeta_j - z_j) = 1.$$

With $w = (w_1, \cdots, w_n)$, the Cauchy–Fantappie kernel $K_w(\zeta, z)$ is given by

$$K_w(\zeta, z) = \sum_{j=1}^{n}(-1)^{j-1}(\frac{H_j}{G})d(\frac{H_1}{G})\wedge\cdots\wedge \widehat{d(\frac{H_j}{G})}\wedge\cdots\wedge d(\frac{H_n}{G})\wedge d\zeta. \tag{15}$$

Exercise 13.3 yields that

$$K_w(\zeta, z) = \frac{1}{G^n}\sum_{j=1}^{n}(-1)^{j-1}H_j dH_1 \wedge \cdots \wedge \widehat{dH_j} \wedge \cdots \wedge dH_n \wedge d\zeta.$$

We put

$$K_j(\zeta, z) = \frac{H_j}{G^n}, \quad j = 1, 2, \cdots, n. \tag{16}$$

Then

$$K_w(\zeta, z) = \sum_{j=1}^{n}(-1)^{j-1}K_j(\zeta, z) \wedge \eta_j(\zeta) \wedge d\zeta,$$

where $\eta_j(\zeta) \equiv dH_1 \wedge \cdots \wedge \widehat{dH_j} \wedge \cdots \wedge dH_n$ is independent of z.

With $w = (w_1, \cdots, w_n)$, where w_j is given by (14), $1 \leq j \leq n$, we put, from now on,

$$K = K_w.$$

We shall deduce Theorem 14.4 from the following three lemmas.

Lemma 14.5. *Let $\phi \in C_0^1(N)$. Fix $z \in N$. Then*

$$\phi(z) = -\frac{(n-1)!}{(2\pi i)^n}\int_N \bar{\partial}\phi(\zeta) \wedge K(\zeta, z).$$

Lemma 14.6. *Let μ be a measure on X with $||\mu|| < \infty$ and $\mu \perp \mathfrak{A}$. Fix z such that*

$$\int_X \frac{d|\mu|(\zeta)}{|\zeta - z|^{2n-1}} < \infty.$$

Then

$$\int_X K_j(\zeta, z)d\mu(\zeta) = 0, \quad j = 1, 2, \ldots, n.$$

Lemma 14.7. *Let μ be a measure on X with $||\mu|| < \infty$ such that $\mu \perp \mathfrak{A}$. Then, for each $\phi \in C_0^1(N)$, we have*

$$\int_X \phi(z)d\mu(z) = 0.$$

Exercise 14.3. The restriction of $C_0^1(N)$ to X is dense in $C(X)$.

Proof of Theorem 14.4. Lemma 14.7 and Exercise 14.3 imply that whenever $\mu \perp \mathfrak{A}$, then $\mu \perp C(X)$. Hence $\mathfrak{A}$ is dense in $C(X)$, and Theorem 14.4 follows. □

We begin the proofs of the three lemmas by establishing certain inequalities for G and H_j, $1 \leq j \leq n$.

Claim. For $\zeta, z \in N$,

$$\operatorname{Re} G(\zeta, z) > 0 \text{ if } \zeta \neq z, \tag{17}$$

$$|G(\zeta, z)| \geq (1-k)|\zeta - z|^2, \tag{18}$$

$$|K_j(\zeta, z)| \leq C \frac{1}{|\zeta - z|^{2n-1}}, \ C \textit{ a constant, and} \tag{19}$$

$$|H_j(\zeta, z)| \leq (1+k)|\zeta - z|. \tag{20}$$

PROOF.

$$H_j(\zeta, z)(\zeta_j - z_j) = |\zeta_j - z_j|^2 + (R_j(\zeta) - R_j(z))(\zeta_j - z_j), \quad \forall j.$$

Hence

$$G(\zeta, z) = \sum_{j=1}^{n} |\zeta_j - z_j|^2 + \sum_{j=1}^{n} (R_j(\zeta) - R_j(z))(\zeta_j - z_j) = |\zeta - z|^2 + B(\zeta, z),$$

where

$$|B(\zeta, z)| \leq |R(\zeta) - R(z)||\zeta - z| \leq k|\zeta - z|^2 < |\zeta - z|^2$$

if $\zeta \neq z$. So

$$G(\zeta, z) = |\zeta - z|^2 + B(\zeta, z) \in \{\operatorname{Re} w > 0\},$$

whence (17) holds. Next,

$$|G(\zeta, z)| \geq |\zeta - z|^2 - k|\zeta - z|^2 = (1-k)|\zeta - z|^2,$$

so (18) holds. Furthermore,

$$|K_j(\zeta, z)| = \frac{|H_j(\zeta, z)|}{|G(\zeta, z)|^n} \leq \frac{|\zeta_j - z_j| + |R_j(\zeta) - R_j(z)|}{|G(\zeta, z)|^n}$$

$$\leq \frac{|\zeta - z| + |R(\zeta) - R(z)|}{(1-k)^n |\zeta - z|^{2n}} \leq \frac{(1+k)|\zeta - z|}{(1-k)^n |\zeta - z|^{2n}} = C \frac{1}{|\zeta - z|^{2n-1}},$$

where $C = (1+k)/(1-k)^n$. This gives (19). □

In a similar way, (12) gives (20). The claim is proved.

PROOF OF LEMMA 14.5. We are given $\phi \in C_0^1(N)$ and a point $z \in N$. We choose a smoothly bounded region D with $\bar{D} \subseteq N$ and $\operatorname{supp}\phi \subseteq D$.

We fix $\epsilon > 0$ and put

$$D_\epsilon = D \setminus \{|\zeta - z| \leq \epsilon\}.$$

Lemma 13.7 then gives

$$(21)\qquad 1 = \frac{(2\pi i)^n}{(n-1)!} \int_{|\zeta - z| = \epsilon} K(\zeta, z).$$

Applying Stokes' Theorem to the smooth $(2n-1)$-form $\phi(\zeta)K(\zeta, z)$ on D_ϵ, we get

$$\int_{\partial D_\epsilon} \phi(\zeta)K(\zeta, z) = \int_{D_\epsilon} d(\phi(\zeta)K(\zeta, z)) = \int_{D_\epsilon} (\bar{\partial}\phi)(\zeta) \wedge K(\zeta, z) \text{ (why?)},$$

and so

$$(22)\qquad \int_{\partial D} \phi(\zeta)K(\zeta, z) - \int_{|\zeta - z| = \epsilon} \phi(\zeta)K(\zeta, z) = \int_{D_\epsilon} (\bar{\partial}\phi)(\zeta) \wedge K(\zeta, z).$$

Also,

$$(23)\qquad \int_{|\zeta - z| = \epsilon} \phi(\zeta)K(\zeta, z)$$
$$= \int_{|\zeta - z| = \epsilon} (\phi(\zeta) - \phi(z))K(\zeta, z) + \int_{|\zeta - z| = \epsilon} \phi(z)K(\zeta, z).$$

EXERCISE 14.4.

$$\lim_{\epsilon \to 0} \int_{|\zeta - z| = \epsilon} (\phi(\zeta) - \phi(z))K(\zeta, z) = 0.$$

Using (21) and Exercise 14.4, we get

$$\lim_{\epsilon \to 0} \int_{|\zeta - z| = \epsilon} \phi(\zeta)K(\zeta, z) = \frac{(n-1)!}{(2\pi i)^n}\phi(z).$$

Letting $\epsilon \to 0$ in (22) then gives

$$\int_{\partial D} \phi(\zeta)K(\zeta, z) - \frac{(n-1)!}{(2\pi i)^n}\phi(z) = \int_D (\bar{\partial}\phi)(\zeta) \wedge K(\zeta, z).$$

Since ∂D lies outside suppϕ, the first integral on the left vanishes, and we get

$$-\frac{(n-1)!}{(2\pi i)^n}\phi(z) = \int_D (\bar{\partial}\phi)(\zeta) \wedge K(\zeta, z) = \int_N (\bar{\partial}\phi)(\zeta) \wedge K(\zeta, z).$$

Thus Lemma (5) is proved. □

EXERCISE 14.5. Each K_j satisfies

$$K_j(\zeta, z) = -K_j(z, \zeta), \quad \zeta, z \in N.$$

EXERCISE 14.6. Let μ be a complex measure on X of finite total mass. Then

$$\int \frac{d|\mu|(\zeta)}{|\zeta - z|^{2n-1}} < \infty$$

for a.a. z in $\mathbb{C}^n$.

PROOF OF LEMMA 14.6. We define

$$S_z = \{G(\zeta, z) : \zeta \in X\}.$$

Then S_z is a compact set in $\mathbb{C}$ and, by (17), S_z lies in the closed half-plane $\{\mathrm{Re}\, w \geq 0\}$. We choose a closed semidisk S such that

$$S_z \subseteq S \subseteq \{\mathrm{Re}\, w \geq 0\}.$$

By Lemma 14.2, there exists a sequence of polynomials $\{P_\nu\}$ satisfying (1) and (2),

$$\lim_{\nu\to\infty} P_\nu(G(\zeta, z)) = \frac{1}{G(\zeta, z)}, \qquad \zeta \in X \setminus \{z\}$$

and

$$|P_\nu(G(\zeta, z))| \leq \frac{C}{|G(\zeta, z)|}, \qquad \zeta \in X \setminus \{z\}, \nu \geq 1.$$

In view of (18), then,

$$|P_\nu(G(\zeta, z))| \leq \frac{C}{(1-k)|\zeta - z|^2}, \qquad \zeta \in X \setminus \{z\}, \nu \geq 1.$$

It follows that for each $\zeta \in X \setminus \{z\}$, and for each j,

$$\lim_{\nu\to\infty} H_j(\zeta, z)(P_\nu(G(\zeta, z)))^n = \frac{H_j(\zeta, z)}{(G(\zeta, z))^n} = K_j(\zeta, z)$$

and

$$|H_j(\zeta, z)||(P_\nu(G(\zeta, z)))^n| \leq |H_j(\zeta, z)|(\frac{C}{1-k})^n \frac{1}{|\zeta - z|^{2n}}$$

$$\leq (1+k)|\zeta - z|(\frac{C}{1-k})^n \frac{1}{|\zeta - z|^{2n}} = \frac{C'}{|\zeta - z|^{2n-1}},$$

where C' is a constant independent of ν.

By hypothesis, $\frac{1}{|\zeta - z|^{2n-1}} \in L^1(|\mu|)$. Thus the sequence $\{H_j P_\nu(G)^n : \nu = 1, 2, \cdots\}$ converges to $K_j(\zeta, z)$ pointwise on $X \setminus \{z\}$, and dominatedly with respect to $|\mu|$. Also, the missing point $\{z\}$ has $|\mu|$-measure 0 (why?). Thus, by the dominated convergence theorem,

$$\int_X H_j(\zeta, z) P_\nu(G(\zeta, z))^n d\mu(\zeta) \to \int_X K_j(\zeta, z) d\mu(\zeta), \tag{24}$$

as $\nu \to \infty$. But

$$H_j(\zeta, z) = (\bar{\zeta}_j + R_j(\zeta)) - (\bar{z}_j + R_j(z));$$

hence $H_j \in \mathfrak{A}$ and so $G \in \mathfrak{A}$, and so $H_j(\zeta, z) P_\nu(G(\zeta, z))^n \in \mathfrak{A}$ for every ν. Hence the left-hand side in (24) vanishes for all ν. Thus $\int_X K_j(\zeta, z) d\mu(\zeta) = 0$, and Lemma 14.6 is proved. $\square$

PROOF OF LEMMA 14.7. Since $\phi \in C_0^1(N)$, for each $z \in N$, Lemma 14.5 gives

$$c_n \phi(z) = -\int_N \bar{\partial}\phi(\zeta) \wedge K(\zeta, z), \tag{25}$$

with $c_n = \frac{(2\pi i)^n}{(n-1)!}$. Hence

$$\begin{aligned} c_n \int_X \phi(z) d\mu(z) &= \int_X \Big[-\int_N \bar{\partial}\phi(\zeta) \wedge K(\zeta, z)\Big] d\mu(z) \\ &= -\int_X \Big[\int_N \sum_{j=1}^n (-1)^{j+1} \bar{\partial}\phi(\zeta) \wedge K_j(\zeta, z) \\ &\qquad \wedge \eta_j(\zeta) \wedge d\zeta \Big] d\mu(z) \\ &= -\sum_{j=1}^n (-1)^{j+1} \int_X d\mu(z) \int_N \bar{\partial}\phi(\zeta) \\ &\qquad \wedge K_j(\zeta, z) \wedge \eta_j(\zeta) \wedge d\zeta. \end{aligned}$$

For each j, $\bar{\partial}\phi(\zeta) \wedge \eta_j(\zeta) \wedge d\zeta$ is a $2n$-form on N. We write it as

$$F_j(\zeta) dx,$$

where F_j is a scalar-valued function and dx is a Lebesgue $2n$-dimensional measure. Then

$$c_n \int_X \phi(z) d\mu(z) = -\sum_{j=1}^n (-1)^{j+1} \int_X d\mu(z) \int_N F_j(\zeta) K_j(\zeta, z) dx. \tag{26}$$

Fix j. By Fubini's Theorem, we have

$$\int_X d\mu(z) \int_N F_j(\zeta) K_j(\zeta, z) dx = \int_N F_j(\zeta) \Big[\int_X K_j(\zeta, z) d\mu(z) \Big] dx. \tag{27}$$

EXERCISE 14.7. Justify this application of Fubini's Theorem.

By Exercise 14.5, $K_j(\zeta, z) = -K_j(z, \zeta)$. Hence, for each $\zeta \in N$,

$$\int_X K_j(\zeta, z) d\mu(z) = -\int_X K_j(z, \zeta) d\mu(z).$$

Lemma 14.6 then gives that

$$\int_X K_j(\zeta, z) d\mu(z) = 0 \tag{28}$$

for a.a. ζ. Since this holds for each j, (26), (27), and (28) yield

$$\int_X \phi(z) d\mu(z) = 0.$$

This proves Lemma 14.7. □

As we saw earlier, Theorem 14.4 follows.

NOTES

The perturbed Stone–Weierstrass theorem in one complex dimension, given in Theorem 14.3, is due to Wermer [We7]. The generalization to higher dimensions, given in Theorem 14.4, was proved under stronger smoothness conditions by Hörmander and Wermer [HöWe]. That proof is based on the theory of approximation on totally real submanifolds in $\mathbb{C}^n$, and is presented in Chapter 17. The proof given in the present chapter, based on certain specially constructed integral kernels, is due to B. Weinstock [Wei].

15

The First Cohomology Group of a Maximal Ideal Space

Given Banach alebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ with maximal ideal spaces $\mathcal{M}_1$ and $\mathcal{M}_2$, if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic as algebras, then $\mathcal{M}_1$ and $\mathcal{M}_2$ are homeomorphic. It is thus to be expected that the topology of $\mathcal{M}(\mathfrak{A})$ is reflected in the algebraic structure of $\mathfrak{A}$, for an arbitrary Banach algebra $\mathfrak{A}$.

One result that we obtained in the direction was this: $\mathcal{M}$ is disconnected if and only if $\mathfrak{A}$ contains a nontrivial idempotent.

We now consider the first Čech cohomology group with integer coefficients. $H^1(\mathcal{M}, Z)$, of a maximal ideal space $\mathcal{M}$.

For decent topological spaces Čech cohomology coincides with singular or simplicial cohomology. We recall the definitions. Let X be a compact Hausdorff space. Fix an open covering $\mathcal{U} = \{U_\alpha\}$ of X, α running over some label set. We construct a simplicial complex as follows: Each U_α is a *vertex*, each pair (U_α, U_β) with $U_\alpha \cap U_\beta \neq \emptyset$ is a 1-*simplex*, and each triple $(U_\alpha, U_\beta, U_\gamma$ with $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ is a 2-*simplex*. A p-cochain ($p = 0, 1, 2$) is a map c^p assigning to each p-simplex an integer, and we require that c^p be an alternating function of its arguments; e.g., $c^1(U_\beta, U_\alpha) = -c^1(U_\alpha, U_\beta)$.

The totality of p-cochains forms a group under addition, denoted $C^p(\mathcal{U})$.

Define the coboundary $\delta : C^p(\mathcal{U}) \to C^{p+1}(\mathcal{U})$ as follows: For $c^0 \in C^0(\mathcal{U})$, (U_α, U_β) a 1-simplex,

$$\delta c^0(U_\alpha, U_\beta) = c^0(U_\beta) - c^0(U_\alpha).$$

For $c^1 \in C^1(\mathcal{U})$, $(U_\alpha, U_\beta, U_\gamma)$ a 2-simplex,

$$\delta c^1(U_\alpha, U_\beta, U_\gamma) = c^1(U_\alpha, U_\beta) + c^1(U_\beta, U_\gamma) + c^1(U_\gamma, U_\alpha).$$

c^1 is a 1-*cocycle* if $\delta c^1 = 0$. The set of all 1-cocycles forms a subgroup $\mathcal{F}^1$ of $C^1(\mathcal{U})$, and $\delta C^0(\mathcal{U})$ is a subgroup of $\mathcal{F}^1$. We define $H^1(\mathcal{U}, Z)$ as the quotient group $\mathcal{F}^1(\mathcal{U})/\delta C^0(\mathcal{U})$. We shall define the cohomology group $H^1(X, Z)$ as the "limit" of $H^1(\mathcal{U}, Z)$ as $\mathcal{U}$ get finer and finer. More precisely

Definition 15.1. Given two coverings $\mathcal{U}$ and $\mathcal{V}$ of X, we say "$\mathcal{V}$ is finer than $\mathcal{U}$" ($\mathcal{V} > \mathcal{U}$) if for each V_α in $\mathcal{V}$ $\exists\phi(\alpha)$ in the label set of $\mathcal{U}$ with $V_\alpha \subset U_{\phi(\alpha)}$.

Note. ϕ is highly nonunique.

Under the relation $>$ the family $\mathcal{F}$ of all coverings of X is a directed set. We have a map

$$\mathcal{U} \to H^1(\mathcal{U}, Z)$$

of this directed set to the family of groups $H^1(\mathcal{U}, Z)$.

For a discussion of direct systems of groups and their application to cohomology we refer the reader to W. Hurewicz and H. Wallman, *Dimension Theory* (Princeton University Press, Princeton, N.J., 1948, Chap. 8, Sec. 4) and shall denote this reference by H.-W.

To each pair $\mathcal{U}$ and $\mathcal{V}$ of coverings of X with $\mathcal{V} > \mathcal{U}$ corresponds for each p a map ρ:

$$C^p(\mathcal{U}) \to C^p(\mathcal{V}),$$

where $\rho c^p(V_{\alpha_0}, V_{\alpha_1}, \ldots, V_{\alpha_p}) = c^p(U_{\phi(\alpha_0)}, \ldots, U_{\phi(\alpha_p)})$, ϕ being as in Definition 15.1.

Lemma 15.1. *ρ induces a homomorphism $K^{\mathcal{U},\mathcal{V}} : H^p(\mathcal{U}, Z) \to H^p(\mathcal{V}, Z)$.*

Lemma 15.2. *$K^{\mathcal{U},\mathcal{V}}$ depends only on $\mathcal{U}$ and $\mathcal{V}$, not on the choice of ϕ.*

For the proofs see H.-W.

The homomorphisms $K^{\mathcal{U},\mathcal{V}}$ make the family $\{H^p(\mathcal{U}, Z)|\mathcal{U}\}$ into a direct system of groups.

Definition 15.2. $H^1(X, Z)$ is the limit group of the direct system of groups $\{H^1(\mathcal{U}, Z)|\mathcal{U}\}$.

$\exists$ a homomorphism $K^{\mathcal{U}} : H^1(\mathcal{U}, Z) \to H^1(X, Z)$ such that for $\mathcal{V} > \mathcal{U}$ we have

$$K^{\mathcal{V}} \circ K^{\mathcal{U},\mathcal{V}} = K^{\mathcal{U}}. \tag{1}$$

(See H.-W.)

Our goal is the following result: Let $\mathfrak{A}$ be a Banach algebra. Put

$$\mathfrak{A}^{-1} = \{x \in \mathfrak{A} | x \text{ has an inverse in } \mathfrak{A}\}$$

and

$$\exp \mathfrak{A} = \{x \in \mathfrak{A} | x = e^y \text{ for some } y \in \mathfrak{A}\}.$$

$\mathfrak{A}^{-1}$ is a group under multiplication and $\exp \mathfrak{A}$ is a subgroup of $\mathfrak{A}^{-1}$.

Theorem 15.3 (Arens-Royden). *Let $\mathcal{M} = \mathcal{M}(\mathfrak{A})$. Then $H^1(\mathcal{M}, Z)$ is isomorphic to the quotient group $\mathfrak{A}^{-1}/\exp \mathfrak{A}$.*

Corollary. *If $H^1(\mathcal{M}, Z) = 0$, then every invertible element x in $\mathfrak{A}$ admits a representation $x = e^y$, $y \in \mathfrak{A}$.*

EXERCISE 15.1. Let $\mathfrak{A} = C(\Gamma)$, Γ the circle. Verify Theorem 15.3 in this case.

EXERCISE 15.2. Do the same for $\mathfrak{A} = C(I)$, I the unit interval.

In the exercises, take as given that $H^1(\Gamma, Z) = Z$ and $H^1(I, Z) = \{0\}$.

Theorem 15.4. *Let X be a compact space. $\exists$ a natural homomorphism*

$$\eta : C(X)^{-1} \to H^1(X, Z)$$

such that η is onto and the kernel of $\eta = \exp C(X)$.

PROOF. Fix $f \in C(X)^{-1}$. Thus $f \neq 0$ on X. We shall associate to f an element of $H^1(X, Z)$, to be denoted $\eta(f)$.

Let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X. A set of functions $g_\alpha \in C(U_\alpha)$ will be called $(f, \mathcal{U})$-*admissible* if

$$f = e^{g_\alpha} \text{ in } U_\alpha \tag{2}$$

and

$$|g_\alpha(x) - g_\alpha(y)| < \pi \qquad \text{for } x, y \text{ in } U_\alpha. \tag{3}$$

Such admissible sets exist whenever $f(U_\alpha)$ lies, for each α, in a small disk excluding 0. Equations (2) and (3) imply that $g_\beta - g_\alpha$ is constant in $U_\alpha \cap U_\beta$.

Now fix a covering $\mathcal{U}$ and an $(f, \mathcal{U})$-admissible set g_α. Then $\exists$ integers $h_{\alpha\beta}$ with

$$\frac{1}{2\pi i}(g_\beta - g_\alpha) = h_{\alpha\beta} \text{ in } U_\alpha \cap U_\beta.$$

The map $h : (U_\alpha, U_\beta) \to h_{\alpha\beta}$ is an element of $C^1(\mathcal{U})$; in fact, h is a 1-cocycle. For given any 1-simplex $(U_\alpha, U_\beta, U_\gamma)$

$$\begin{aligned} \delta h(U_\alpha, U_\beta, U_\gamma) &= h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} \\ &= \frac{1}{2\pi i}\{g_\beta - g_\alpha + g_\gamma - g_\beta + g_\alpha - g_\gamma\} = 0 \end{aligned}$$

at each point of $U_\alpha \cap U_\beta \cap U_\gamma$.

Denote by $[h]$ the cohomology class of h in $H^1(\mathcal{U}, Z)$.

(4) $[h]$ is independent of the choice of $\{g_\alpha\}$ and depends only on f and $\mathcal{U}$.

For let $\{g'_\alpha\}$ be another $(f, \mathcal{U})$-admissible set. By (2) and (3), $\exists k_\alpha \in Z$ with

$$g'_\alpha(x) - g_\alpha(x) = 2\pi i k_\alpha \qquad \text{for } x \in U_\alpha.$$

The cocylce h' determined by $\{g'_\alpha\}$ is given by

$$h'_{\alpha\beta} = h'(U_\alpha, U_\beta) = \frac{1}{2\pi i}(g'_\beta(x) - g'_\alpha(x))$$

$(x \in U_\alpha \cap U_\beta)$. Hence

$$h'_{\alpha\beta} = h_{\alpha\beta} + \delta k,$$

where k is the 0-cochain in $C^0(\mathcal{U})$ defined by $k(U_\alpha) = k_\alpha$. Thus $[h'] = [h]$, as desired.

We define

$$\eta_{\mathcal{U}(f)} = [h]$$

and

$$\eta(f) = K^{\mathcal{U}}([h]) \in H^1(X, Z).$$

Using (1) we can verify that $\eta(f)$ depends only on f, not on the choice of the covering $\mathcal{U}$.

(5) η maps $C(X)^{-1}$ onto $H^1(X, Z)$.

To prove this fix $\xi \in H^1(X, Z)$. Choose a covering $\mathcal{U}$ and a cocycle h in $C^1(\mathcal{U})$ with $K^{\mathcal{U}}([h]) = \xi$. Put $h_{\nu\mu} = h(U_\nu, U_\mu)$. Since X is compact and so an arbitrary open covering admits a finite covering finer than itself, we may assume that $\mathcal{U}$ is finite, $\mathcal{U} = \{U - 1, U_2, \ldots, U_s\}$.

Choose a partition of unity χ_α, $1 \leq \alpha \leq s$, with supp $\chi_\alpha \subset U_\alpha$, $\chi_\alpha \in C(X)$, and $\sum_{\alpha=1}^{s} \chi_\alpha = 1$. For each k define

$$g_k = 2\pi i \sum_{\nu=1}^{s} h_{\nu k} \chi_\nu(x) \qquad \text{for } x \in U_k,$$

where we put $h_{\nu k} = 0$ unless U_ν meets U_k. Then $g_k \in C(U_k)$. Fix $x \in U_j \cap U_k$. Note that unless U_ν meets $U_k \cap U_j$, $\chi_\nu(x) = 0$. Then

$$(g_k - g_j)(x) = 2\pi i \sum_{\nu} \chi_\nu(x)(h_{\nu k} - h_{\nu j}).$$

Since h is a 1-cocyle, $h_{k\nu} + h_{\nu j} + h_{jk} = 0$ whenever $U_k \cap U_\nu \cap U_j \neq \emptyset$. Hence in $U_j \cap U_k$,

$$g_k - g_j = 2\pi i \sum_{\nu} x_\nu h_{jk} = 2\pi i h_{jk}.$$

Define f_α in U_α by $f_\alpha = e^{g_\alpha}$. Then $f_\alpha \in C(U_\alpha)$ and in $U_\alpha \cap U_\beta$,

$$\frac{f_\beta}{f_\alpha} = e^{g_\beta - g_\alpha} = e^{2\pi i h_{\alpha\beta}} = 1.$$

Thus $f_\alpha = f_\beta$ in $U_\alpha \cap U_\beta$, so the different f_α fit together to a single function f in $C(X)$. Also,

$$f_\alpha = e^{g_\alpha} \text{ in } U_\alpha \qquad \text{and} \qquad g_\beta - g_\alpha = 2\pi i h_{\alpha\beta} \text{ in } U_\alpha \cap U_\beta.$$

From this and the definition of η, we can verify that $\eta(f) = K^{\mathcal{U}}([h]) = \xi$.

(6) Fix f in the kernel of η. Then $f \in \exp C(X)$.

For $\eta(f)$ is the zero element of $H^1(X, Z)$. Hence $\exists$ covering $\mathcal{V}$ such that if h is the cocycle in $C^1(\mathcal{V})$ associated to f by our construction, then the cohomology class of h is 0; i.e., if

$$f = e^{g_\alpha} \text{ in } U_\alpha,$$

then $\exists H \in C^0(\mathcal{V})$ such that

$$g_\beta - g_\alpha = 2\pi i(H_\beta - H_\alpha) \text{ in } V_\alpha \cap V_\beta.$$

Then

$$g_\beta - 2\pi H_\beta = g_\alpha - 2\pi H_\alpha \text{ in } V_\alpha \cap V_\beta.$$

Hence $\exists$ global function G in $C(X)$ with $G = g_\alpha - 2\pi H_\alpha$ in V_α for each α. Then $f = e^G$, and we are done.

Since it is clear that η vanishes on exp $C(X)$, the proof of Theorem 15.4 is complete. □

Note. We leave to the reader to verify that η is natural.

Now let X be a compact space and $\mathcal{L}$ a subalgebra of $C(X)$. The map η (of Theorem 15.4) restricts to $\mathcal{L}^{-1} = \{f \in \mathcal{L} | 1/f \in \mathcal{L}\}$, mapping $^{-\infty}$ into $H^1(X, Z)$.

Definition 15.3. $\mathcal{L}$ is *full* if
(a) η maps $\mathcal{L}^{-1}$ onto $H^1(X, Z)$.
(b) $x \in \mathcal{L}^{-1}$ and $\eta(x) = 0$ imply $\exists y \in \mathcal{L}$, with $x = e^y$.

Next let X be a compact polynomially convex subset of $\mathbb{C}^n$.

Definition 15.4. $\mathcal{H}(X) = \{f \in C(X) | \exists$ neighborhood of U of X and $\exists F \in H(U)$ with $F = f$ on $X\}$.

$\mathcal{H}(X)$ is a subalgebra of $C(X)$.

Lemma 15.5. *$\mathcal{H}(X)$ is full.*

PROOF. Fix $\gamma \in H^1(X, Z)$. Then $\exists$ a covering $\mathcal{U}$ of X and a cocycle $h \in C^1(\mathcal{U})$ with $K^{\mathcal{U}}([h]) = \gamma$.

Without loss of generality, we may assume that

$$\mathcal{U} = \{U_\alpha \cap X | 1 \leq \alpha \leq s\}, \text{ each } U_\alpha \text{ open in } \mathbb{C}^n.$$

(Why?)

For each α choose $\xi_\alpha \in C_0^\infty(U_\alpha)$, with $\sum_{\alpha=1}^s \xi_\alpha = 1$ in some neighborhood N of X. Put $h_{\alpha\beta} = h(U_\alpha \cap X, U_\beta \cap X) \in Z$. Fix α and put for $x \in U_\alpha$,

$$g_\alpha(x) = 2\pi \sum_{\nu=1}^s h_{\nu\alpha} \xi_\nu(x),$$

where $h_{\nu\alpha} = 0$ unless $U_\nu \cap U_\alpha \neq \emptyset$. Then $g_\alpha \in C^\infty(U_\alpha)$, and, as in the proof of Theorem 15.4, we have in $U_\alpha \cap U_\beta \cap N$,

$$g_\beta - g_\alpha = 2\pi i h_{\alpha\beta}. \tag{7}$$

Hence $\bar{\partial} g_\beta - \bar{\partial} g_\alpha = 0$ in $U_\alpha \cap U_\beta \cap N$, so the $\bar{\partial} g_\alpha$ fit together to a $\bar{\partial}$-closed (0, 1)-form defined in N.

By Lemma 7.4 $\exists$ a p-polyhedron $\prod$ with $X \subset \prod \subset N$. By Theorem 7.6 $\exists$ a neighborhood W of $\prod$ and $u \in C^\infty(W)$ with

$$\bar{\partial} u = \bar{\partial} g_\alpha \text{ in } W \cap U_\alpha. \tag{8}$$

Put $V_\alpha = U_\alpha \cap W$, $1 \leq \alpha \leq s$. Then $\mathcal{U} = \{V_\alpha \cap X | 1 \leq \alpha \leq s\}$.

Put $g'_\alpha = g_\alpha - u$ in V_α. Then $g'_\alpha \in H(V_\alpha)$, by (8). Also, by (7),

$$\frac{1}{2\pi}(g'_\beta - g'_\alpha) = \frac{1}{2\pi}(g_\beta - g_\alpha) = h_{\alpha\beta} \text{ in } V_\alpha \cap V_\beta.$$

Define $f = e^{g_\alpha}$ in V_α for each α. In $V_\alpha \cap V_\beta$ the two definitions of f are

$$e^{g'_\alpha} \text{ and } e^{g'_\beta} = e^{g'_\alpha + 2\pi h_{\alpha\beta}} = e^{g'_\alpha}.$$

Hence f is well defined in $\bigcup_\alpha V_\alpha$ and holomorphic there, so $f|_x \in \mathcal{H}(X)$ and, in fact, $\in (\mathcal{H}(X))^{-1}$.

$g'_\beta - g'_\alpha = 2\pi i h_{\alpha\beta}$, whence $\eta(f) = K^{\mathcal{U}}([h]) = \gamma$. We have verified (a) in Definition 15.3.

Now fix $f \in (\mathcal{H}(X))^{-1}$ with $\eta(f) = 0$. Let F be holomorphic in a neighborhood of X with $F = f$ on X.

Since $\eta(f) = 0$, $\eta_{\mathcal{U}(f)} = 0$ for some covering $\mathcal{U}$. Choose a covering of X by open subsets W_α of $\mathbb{C}^n$, $1 \leq \alpha \leq s$, such that

$$\mathcal{W} > \mathcal{U}. \tag{9}$$

$$\exists G_\alpha \in H(W_\alpha) \qquad \text{with } F = e^{G_\alpha} \text{ in } W_\alpha. \tag{10}$$

$$|G_\alpha(x) - G_\alpha(y)| < \pi \qquad \text{for } x, y \in W_\alpha. \tag{11}$$

$$\text{If } W_\alpha \cap W_\beta \neq \emptyset, \text{ then } W_\alpha \cap W_\beta \text{ meets } X. \tag{12}$$

Let $\mathcal{W} = \{W_\alpha \cap X | 1 \leq \alpha \leq s\}$. $\eta_{\mathcal{U}(f)} = 0$, so $\eta_{\mathcal{W}(f)} = 0$. Hence $\exists$ integers k_α such that if $(W_\alpha \cap X) \cap (W_\beta \cap X) \neq \emptyset$, then in $(W_\alpha \cap X) \cap (W_\beta \cap X)$,

$$\frac{1}{2\pi}(G_\beta - G_\alpha) = k_\beta - k_\alpha. \tag{13}$$

Now fix α and β with $W_\alpha \cap W_\beta \neq \emptyset$. By (12), $(W_\alpha \cap X) \cap (W_\beta \cap X) \neq \emptyset$. Hence, by (13),

$$G_\beta - G_\alpha = 2\pi k_\beta - 2\pi i k_\alpha \text{ in } W_\alpha \cap W_\beta \cap X.$$

Also, because of (10) and (11),

$$G_\beta - G_\alpha \text{ is constant in } W_\alpha \cap W_\beta.$$

Hence

$$G_\beta - G_\alpha = 2\pi i k_\beta - 2\pi i k_\alpha \text{ in } W_\alpha \cap W_\beta$$

or

$$G_\beta - 2\pi i k_\beta = g_\alpha - 2\pi i k_\alpha \text{ in } W_\alpha \cap W_\beta.$$

Hence $\exists G \in H(\bigcup_\alpha W_\alpha)$ with $G = G_\alpha - 2\pi i k_\alpha$ in W_α for each α. Then

$$F = e^G \text{ in } \bigcup_\alpha W_\alpha.$$

Since $G|_x \in \mathcal{H}(X)$, we have verified (*b*) in Definition 15.2. So the lemma is proved. $\square$

Lemma 15.6. *Let $\mathcal{L}$ be a finitely generated uniform algebra on a space X with $X = \mathcal{M}(\mathcal{L})$. Then $\mathcal{L}$ is full [as subalgebra of $C(X)$.]*

PROOF. By Exercise 7.3 it suffices to assume that $\mathcal{L} = P(X)$, X a compact polynomially convex set in $\mathbb{C}^n$.

By the Oka-Weil theorem $\mathcal{H}(X) \subset P(X)$. Fix $\gamma \in H^1(X, Z)$. By the last lemma, $\exists f \in (\mathcal{H}(X))^{-1}$ with $\eta(f) = \gamma$. Then $f \in (P(X))^{-1}$. Thus η maps $(P(X))^{-1}$ onto $H^1(X, Z)$. Now fix $f \in (P(X))^{-1}$ with $\eta(f) = 0$, and fix $\varepsilon > 0$. Choose a polynomial g with

$$||g - f|| < \varepsilon < \inf_x |f|,$$

the norm being taken in $P(X)$. Put $h = (f - g)/f$. Then $||h|| < 1$ and $g = f(1 - h)$. Hence $1 - h \in \exp C(X)$ (why?) and so $\eta(1 - h) = 0$. Hence

$$\eta(g) = \eta(f) = 0.$$

But $g \in (\mathcal{H}(X))^{-1}$, whence by the last lemma $\exists g^\circ \in \mathcal{H}(X)$ with $g = e^{g^\circ}$.

Also, $1 - h = e^k$ for some $k \in P(X)$, since $||h|| < 1$. (Why?) Hence $f = e^{g^\circ - k}$, so $f \in \exp(P(X))$. Thus $P(X)$ is full. $\square$

To extend this result to a uniform algebra $\mathfrak{A}$ that fails to be finitely generated, we may express $\mathfrak{A}$ as a "limit" of its finitely generated subalgebras. For this extension we refer the reader to H. Royden, Function algebras, *Bull. Am. Math. Soc.* **69** (1963), 281-298. The following is proved there (Proposition 11):

Lemma 15.7. *Let $\mathcal{L}$ be an arbitrary uniform algebra on a space X with $X = \mathcal{M}(\mathcal{L})$. Then $\mathcal{L}$ is full.*

PROOF OF THEOREM 15.3. Put $X = \mathcal{M}$ and let $\mathcal{L}$ be the uniform closure of $\mathfrak{A}$ on X. Then $X = \mathcal{M}(\mathcal{L})$. By Lemma 15.7 $\mathcal{L}$ is full [as subalgebra of $C(X)$].

Let $x \in \mathfrak{A}^{-1}$. Then $\bar{x} \in (C(X))^{-1}$. Define a map Φ of $\mathfrak{A}^{-1}$ into $H^1(X, Z)$ by

$$\Phi(x) = \eta(\bar{x}).$$

We claim Φ is onto $H^1(X, Z)$. Fix $\gamma \in H^1(X, Z)$. Since $\mathcal{L}$ is full, $\exists f \in \mathcal{L}^{-1}$ with $\eta(f) = \gamma$. Choose $\varepsilon > 0$ with $\inf_x |f| > \varepsilon$, and choose $g \in \mathfrak{A}$ with $|\hat{g} - f| < \varepsilon$ on X. Then $g \in \mathfrak{A}^{-1}$, $\hat{g} = f(1 - (f - \hat{g})/f)$, and $\sup_x |(f - \hat{g})/f| < 1$. Hence $\exists b \in C(X)$ with $1 - (f - \hat{g})/f = e^b$, and so $\eta(\hat{g}) = \eta(f) = \gamma$. Thus $\Phi(g) = \gamma$, so Φ is onto, as claimed.

Next we claim that the kernel of $\Phi = \exp \mathfrak{A}$. Since one direction is clear, it remains to show that $x \in \mathfrak{A}^{-1}$ and $\Phi(x) = 0$ implies that $x \in \exp \mathfrak{A}$.

Then fix $x \in \mathfrak{A}^{-1}$ with $\Phi(x) = \eta(\hat{g}) = 0$. Since $\mathcal{L}$ is full and $\hat{x} \in \mathcal{L}^{-1}$, $\exists F \in \mathcal{L}$ with $\hat{x} = d^F$. Since F is in the uniform closure of $\mathfrak{A}$, e^F is in the uniform closure of functions e^h, $h \in \mathfrak{A}$.

Hence $\exists g = e^h$ with $h \in \mathfrak{A}$ and

$$|\hat{x} - \hat{g}| < \frac{1}{3} \inf_X |\hat{x}| \text{ on } X.$$

Then

$$|\hat{g}| > \tfrac{2}{3} \inf_X |\hat{x}|, \qquad \text{so } \frac{1}{|\hat{g}|} < \frac{3}{2} \cdot \frac{1}{\inf_x |\hat{x}|}.$$

Hence uniformly on X,

$$|1 - \hat{x}\hat{g}^{-1}| = |\hat{x} - \hat{g}| \cdot |\hat{g}^{-1}| < \frac{1}{2}.$$

It follows that for large n, $||(1 - xg^{-1})^n||^{1/n} < \frac{3}{4}$, and so the series

$$-\sum_1^\infty \frac{1}{n}(1 - xg^{-1})^n$$

converges in $\mathfrak{A}$ to an element k. Since

$$\log(1 - z) = -\sum_1^\infty \frac{1}{n} z^n, \qquad |z| < 1,$$

$k = \log(xg^{-1})$, so that $xg^{-1} = e^k$. Hence $x = e^{k+h} \in \exp \mathfrak{A}$. Hence the kernel of Φ is $\exp \mathfrak{A}$, as claimed.

Φ thus induces an isomorphism of $\mathfrak{A}^{-1}/\exp \mathfrak{A}$ onto $H^1(X, Z)$, and Theorem 15.3 is proved. □

Note. No analogous algebraic interpretation of the higher cohomology groups $H^p(\mathcal{M}, Z)$, $p > 1$, has so far been obtained. However, one has the following result:

Theorem 15.8. *Let $\mathfrak{A}$ be a Banach algebra with n generators. Then $H^p(\mathcal{M}, \mathbb{C}) = 0$, $p \geq n$.*

This result is due to A. Browder, Cohomology of maximal ideal spaces, *Bull. Am. Math. Soc.* **67** (1961), 515-516. Observe that if $\mathfrak{A}$ has n generators, then $\mathcal{M}$ is homeomorphic to a subset of $\mathbb{C}^n$ and hence that the vanishing of $H^p(\mathcal{M}, \mathbb{C})$ is obvious for $p \geq 2n$.

NOTES

For the first theorem of the type studied in this section (Theorem 15.4) see S. Eilenberg, Transformations continues en circonférence et la topologie du plan, *Fund. Math.* **26** (1936) and N. Bruschlinsky, Stetige Abbildungen und Bettische Gruppen der Dimensionszahl 1 und 3, *Math. Ann.* **109** (1934). Theorem 15.3 is due to R. Arens, The group of invertible elements of a commutative Banach algebra, *Studia Math.* **1** (1963), and H. Royden, Function algebras, *Bull. Am. Math. Soc.* **69** (1963). The proof we have given follows Royden's paper.

16

The $\bar{\partial}$-Operator in Smoothly Bounded Domains

Let Ω be a bounded open subset of $\mathbb{C}^n$. We are essentially concerned with the following problem: Given a form f of type (0, 1) on Ω with $\bar{\partial} f = 0$, find a function u on Ω such that $\bar{\partial}_u = f$.

In order to be able to use the properties of operators on Hilbert space in attacking this question, we shall consider L^2-spaces rather than (as before) spaces of smooth functions.

$L^2(\Omega)$ denotes the space of measurable functions u on Ω with $\int_\Omega |u|^2 dV < \infty$, where dV is Lebesgue measure.

$L^2_{0,1}(\Omega)$ is the space of (0, 1)-forms

$$f = \sum_{j=1}^{n} f_j d\bar{z}_j,$$

where each $f_j \in L^2(\Omega)$. Put $|f|^2 = \sum_{j=1}^{n} |f_j|^2$. Analogously, $L^2_{0,2}(\Omega)$ is the space of (0, 2)-forms

$$\phi = \sum_{i<j} \phi_{ij} d\bar{z}_i \wedge d\bar{z}_j,$$

where each $\phi_{ij} \in L^2(\Omega)$.

We shall define an operator T_0 from a subspace of $L^2(\Omega)$ to $L^2_{0,1}(\Omega)$ such that T_0 coincides with $\bar{\partial}$ on functions that are smooth on $\bar{\Omega}$.

Definition 16.1. Let $u \in L^2(\Omega)$. Fix $k \in L^2(\Omega)$ and fix j, $1 \leq j \leq n$. We say

$$\frac{\partial u}{\partial \bar{z}_j} = k$$

if for all $g \in C_0^\infty(\Omega)$ we have

$$-\int_\Omega u \frac{\partial g}{a\bar{z}_j} dV = \int_\Omega gk dV.$$

Note. Thus $k = \partial u/\partial \bar{z}_j$ in the sense of the theory of distributions. If u is smooth on $\bar{\Omega}$, then $k = \partial u/\partial \bar{z}_j$, in the usual sense.

Definition 16.2.

$$\mathcal{D}_{T_0} = \left\{ u \in L^2(\Omega) | \text{ for each } j,\ 1 \le j \le n \exists k_j \in l^2(\Omega) \text{ with } \frac{\partial u}{\partial \bar{z}_j} = k_j \right\}.$$

For $u \in \mathcal{D}_{T_0}$,

$$T_0 u = \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j \in L^2_{0,1}(\Omega).$$

Fix $\sum_{j=1}^{n} f_j d\bar{z}_j$ with each $f_j \in L^2(\Omega)$. $\partial f_j/\partial \bar{z}_k$ and $\partial f_k/\partial \bar{z}_j$ are defined as distributions.

Definition 16.3.

$$\mathcal{D}_{S_0} = \left\{ f = \sum_{j=1}^{n} f_j d\bar{z}_j \in L^2_{0,1} \mid | \frac{\partial f_j}{\partial \bar{z}_k} - \frac{\partial f_k}{\partial \bar{z}_j} \in L^2(\Omega), \text{ all } j, k \right\}.$$

For $f \in \mathcal{D}_{S_0}$,

$$S_0 f = \sum_{j<k} \left(\frac{\partial f_j}{\partial \bar{z}_k} - \frac{\partial f_k}{\partial \bar{z}_j} \right) d\bar{z}_k \wedge d\bar{z}_j \in L^2_{0,2}(\Omega).$$

Note that S_0 coincides with $\bar{\partial}$ on smooth forms f. Note also that if $u \in \mathcal{D}_{T_0}$, then $T_0 u \in \mathcal{D}_{S_0}$, and

(1) $$S_0 \cdot T_0 = 0.$$

Now let Ω be defined by the inequality $\rho < 0$, where ρ is a smooth real-valued function in some neighborhood of $\overline{\Omega}$. Assume that the gradient of $\rho \neq 0$ on $\partial\Omega$. We impose on ρ the following condition:

(2) For all $z \in \partial\Omega$, if $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$ and $\sum_j \partial\rho/\partial z_j(z)\xi_j = 0$,

then

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z)\xi_j\bar{\xi}_k \ge 0.$$

Theorem 16.1. *Let ρ satisfy condition (2). For every $g \in L^2_{0,1}(\Omega)$ with $S_0 g = 0$, $\exists u \in \mathcal{D}_{T_0}$ such that*
(a) $T_0 u = g$, and
(b) $\int_\Omega |u|^2 dV \le e^{R^2} \cdot \int_\Omega |g|^2 dV$,
if $\Omega \subset \{z \in \mathbb{C}^n \big| |z| \le R\}$.

We need some general results about linear operators on Hilbert space.

Let H_1 and H_2 be Hilbert spaces, and let A be a linear transformation from a dense subspace $\mathcal{D}_A$ of H_1 into H_2.

Definition 16.4. A is *closed* if for each sequence $g_n \in \mathcal{D}_A$,

$$g_n \to g \qquad \text{and} \qquad Ag_n \to h$$

implies that $g \in \mathcal{D}_A$ and $Ag = h$.

Definition 16.5.

$$\mathcal{D}_{A^*} = \{x \in H_2 | \exists x^* \in H_1 \text{ with } (Au, x) = (u, x^*) \text{ for all } u \in \mathcal{D}_A.\}$$

Since $\mathcal{D}_A$ is dense, x^* is unique if it exists. For $x \in \mathcal{D}_A^*$, define $A^*x = x^*$. A^* is called the *adjoint* of A. $\mathcal{D}_{A^*}$ is a linear space and A^* is a linear transformation of $\mathcal{D}_{A^*} \to H_1$.

Proposition. *If A is closed, then $\mathcal{D}_{A^*}$ is dense in H_2. Moreover, if $\beta \in H_1$ and if for some constant δ*

$$|(A^* f, \beta)| \leq \delta ||f||$$

for all $f \in \mathcal{D}_{A^*}$, then $\beta \in \mathcal{D}_A$.

For the proof of this proposition and related matters the reader may consult, e.g., F. Riesz and B. Sz.-Nagy, *Lecons d'analyse fonctionelle*, Budapest, 1953, Chap. 8.

Consider now three Hilbert spaces H_1, H_2, and H_3 and densely defined and closed linear operators

$$T : H_1 \to H_2 \qquad \text{and} \qquad S : H_2 \to H_3.$$

Assume that

$$S \cdot T = 0; \tag{3}$$

i.e., for $f \in \mathcal{D}_T$, $Tf \in \mathcal{D}_S$ and $S(Tf) = 0$.

We write $(u, v)_j$ for the inner product of u and v in H_j, $j = 1, 2, 3$, and similarly $||u||_j$ for the norm in H_j.

Theorem 16.2. *Assume $\exists$ a constant c such that for all $f \in \mathcal{D}_{T^*} \cap \mathcal{D}_S$,*

$$||T^* f||_1^2 + ||Sf||_3^2 \geq c^2 ||f||_2^2. \tag{$*$}$$

Then if $g \in H_2$ with $Sg = 0$, $\exists u \in \mathcal{D}_T$ such that

$$Tu = g \tag{4}$$

and

$$||u||_1 \leq \frac{1}{c} ||g||_2. \tag{5}$$

PROOF. Put $N_S = \{h \in \mathcal{D}_S | Sh = 0\}$. N_S is a closed subspace of H_2. (Why?)

We claim that if $g \in N_S$, then

$$|(g, f)_2| \leq \frac{1}{c} ||T^* f||_1 \cdot ||g||_2, \tag{6}$$

for all $f \in \mathcal{D}_T$

To show this, fix $f \in \mathcal{D}_{T^*}$.

$$f = f' + f'', \text{ where } f' \perp N_S, \, f'' \in N_S.$$

By (*) we have $||T^* f''||_1 \geq c||f''||_2$. Then

$$|(f, g)_2| = |(f'', g)_2| \leq ||g||_2 \cdot ||f''||_2 \leq \frac{1}{c}||g||_2 \cdot ||T^* f''||_1.$$

But $T^* f' = 0$, for if $h \in \mathcal{D}_T$, $(Th, f') = (h, T^* f')$ and the left-hand side $= 0$, because $f' \perp N_S$ while $S(Th) = 0$ by (3). Hence $T^* f = T^* f''$, and so (6) holds, as claimed.

We now define a linear functional L on the range of T^* in H_1 by

$$L(T^* f) = (f, g)_2, \; f \in \mathcal{D}_{T^*}, \; g \text{ fixed in } N_S.$$

By (6), then,

$$|L(T^* f)| \leq \frac{1}{c}||g||_2||T^* f||_1.$$

It follows that L is well defined on the range of T^* and that $||L|| \leq (1/c)||g||_2$. Hence $\exists u \in H_1$ representing L; i.e.,

$$L(T^* f) = (T^* f, u)_1,$$

and $||u||_1 = ||L||$. It follows by the proposition that $u \in \mathcal{D}_T$, and

$$(f, g)_2 = (T^* f, u)_1 = (f, Tu)_2,$$

all $f \in \mathcal{D}_{T^*}$.

Hence $g = Tu$, and $||u||_1 \leq (1/c)||g||_2$. Thus (4) and (5) are established. □

It is now our task to verify hypothesis (*) for our operators T_0 and S_0 in order to apply Theorem 16.2 to the proof of Theorem 16.1. This means that we must find a lower bound for $||T_0^* f||^2 + ||S_0 f||^2$. For this purpose it is advantageous to use not the usual inner product on $L^2(\Omega)$ but an equivalent inner product based on a weight function.

Let ϕ be a smooth positive function defined in a neighborhood of $\bar{\Omega}$. Put $H_1 = L^2(\Omega)$ with the inner product

$$(f, g)_1 = \int_\Omega f\bar{g}e^{-\phi}dV.$$

Similarly, let H_2 be the Hilbert space obtained by imposing on $L^2_{0,1}(\Omega)$ the inner product

$$\left(\sum_{j=1}^n f_j d\bar{z}_j, \sum_{j=1}^n g_j d\bar{z}_j\right)_2 = \int_\Omega \left(\sum_{j=1}^n f_j \bar{g}_j\right) e^{-\phi}dV.$$

Finally define H_3 in an analogous way by putting a new inner product on $L^2_{0,2}(\Omega)$. Then

$$T_0 : H_1 \to H_2, \qquad S_0 : H_2 \to H_3.$$

It is easy to verify that $\mathcal{D}_{T_0}$, $\mathcal{D}_{S_0}$ are dense subspaces of H_1 and H_2, respectively, and that T_0 and S_0 are closed operators. Our basic result is the following: Define $C^1_{0,1}(\bar\Omega) = \{f = \sum_{j=1}^n f_j d\bar z_j \mid \text{each } f_j \in C^1 \text{ in a neighborhood of } \bar\Omega.\}$

Theorem 16.3. *Fix f in $C^1_{0,1}(\bar\Omega)$. Let $f \in \mathcal{D}_{T_0^*} \cap \mathcal{D}_{S_0}$. Then*

$$(7) \quad ||T_0^* f||_1^2 + ||S_0 f||_3^2 = \sum_{j,k} \int_\Omega f_j \bar f_k \frac{\partial^2\phi}{\partial z_j \partial \bar z_k} e^{-\phi} dV$$

$$+ \sum_{j,k} \int_\Omega \left| \frac{\partial f_k}{\partial \bar z_j} \right|^2 e^{-\phi} dV + \sum_{j,k} \int_{\partial\Omega} f_j \bar f_k \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} e^{-\phi} dS,$$

dS denoting the element of surface area on $\partial\Omega$.

Suppose for the moment that Theorem 16.3 has been established. Put

$$\phi(z) = \sum_{j=1}^n |z_j|^2 = |z|^2.$$

Then $\partial^2\phi/\partial z_j \partial \bar z_k = 0$ if $j \neq k$, $= 1$ if $j = k$. The first integral on the right in (7) is now

$$\sum_{j=1}^n \int_\Omega |f_j|^2 e^{-\phi} dV = ||f||_2^2.$$

The second integral is evidently ≥ 0. Now

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar z_k} f_j \bar f_k \geq 0 \qquad \text{if } \sum_j \frac{\partial \rho}{\partial z_j} f_j = 0 \text{ on } \partial\Omega,$$

by (2). Hence (7) gives

$$(8) \qquad ||T_0^* f||_1^2 + ||S_0 f||_3^2 \geq ||f||_2^2,$$

if

$$(9) \qquad \sum_j \frac{\partial \rho}{\partial z_j} f_j = 0 \text{ on } \partial\Omega.$$

We shall show below that (9) holds whenever $f \in \mathcal{D}_{T_0^*} \cap \mathcal{D}_{S_0}$ and f is C^1 in a neighborhood of $\bar\Omega$. Thus Theorem 16.3 implies that (8) hold for each smooth f in $\mathcal{D}_{T_0^*} \cap \mathcal{D}_{S_0}$.

We now quote a result from the theory of partial differential operators which seems plausible and is rather technical. We refer for its proof to [39], Proposition 2.1.1.

Proposition. *Let $f \in \mathcal{D}_{T_0^*} \cap \mathcal{D}_{S_0}$ (with no smoothness assumptions). Then $\exists$a sequence $\{f_n\}$ with $f_n \in \mathcal{D}_{T_0^*} \cap \mathcal{D}_{S_0}$ and f_n in C^1 in a neighborhood of $\bar{\Omega}$ such that as $n \to \infty$,*

$$||f_n - f||_2 \to 0, \qquad ||T_0^* f_n - T_0^* f||_1 \to 0, \qquad ||S_0 f_n - S_0 f||_3 \to 0.$$

Since (8) holds when f is smooth, the proposition gives that (8) holds for all $f \in \mathcal{D}_{T_0^*} \cap \mathcal{D}_{S_0}$.

Theorem 16.2 now applies to T_0 and S_0 with $c = 1$. It follows from (4) and (5) that if $g = \sum_{j=1}^n g_j d\bar{z}_j \in H_2$, and if $S_0 g = 0$, then $\exists u$ in H_1 with $T_0 u = g$ and $||u||_1 \leq ||g||_2$. Thus

$$\int_\Omega |u|^2 e^{-\phi} dV \leq \int_\Omega |g|^2 e^{-\phi} dV.$$

Now if $\Omega \subset \{z \in \mathbb{C}^n \,|\, |z| \leq R\}$, then

$$\begin{aligned}
\int_\Omega |u|^2 dV &= \int_\Omega |u|^2 e^{-\phi} \cdot e^{\phi} dV \\
&\leq \int_\Omega |u|^2 e^{-\phi} \cdot e^{R^2} dV \leq e^{R^2} \int_\Omega |g|^2 e^{-\phi} dV \\
&\leq e^{R^2} \int_\Omega |g|^2 dV,
\end{aligned}$$

and so (b) holds. Thus Theorem 16.1 follows form Theorem 16.3.

From now on ρ is assumed to satisfy (2) and Ω is defined by $\rho < 0$. We also shall write T and S instead of T_0 and S_0. Let us now begin the proof of (7).

Lemma 16.4. *Let $f = \sum_{j=1}^n f_j d\bar{z}_j \in C^1_{0,1}(\bar{\Omega})$. If $f \in \mathcal{D}_{T^*}$, then*

$$\sum_{j=1}^n f_j \frac{\partial \rho}{\partial z_j} = 0 \text{ on } \partial\Omega, \tag{9}$$

and

$$T^* f = -\sum_{j=1}^n e^{\phi} \frac{\partial}{\partial z_j} (f_j e^{-\phi}). \tag{10}$$

Proof. Let h be a function in C^2 in a neighborhood of $\bar{\Omega}$ and $h > 0$. Put $R = h \cdot \rho$.

Fix $z \in \partial\Omega$ and choose $(\xi_1, \ldots, \xi_n)$ satisfying

$$\sum_{j=1}^n \frac{\partial R}{\partial z_j}(z) \xi_j = 0. \tag{11}$$

Then at z,

$$\frac{\partial^2 R}{\partial z_j \partial \bar{z}_k} = \frac{\partial}{\partial \bar{z}_k}\left(h\frac{\partial \rho}{\partial z_j} + \frac{\partial h}{\partial z_j}\rho\right)$$
$$= \frac{\partial h}{\partial \bar{z}_k}\frac{\partial \rho}{\partial z_j} + h\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} + \frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}\rho + \frac{\partial h}{\partial z_j}\frac{\partial \rho}{\partial \bar{z}_k}.$$

Hence

$$\sum_{j,k}\frac{\partial^2 R}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k = \left(\sum_k \frac{\partial h}{\partial \bar{z}_k}\bar{\xi}_k\right)\left(\sum_j \frac{\partial \rho}{\partial z_j}\xi_j\right)$$
$$+ h\sum_{j,k}\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k + \rho\sum\frac{\partial^2 h}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k$$
$$+ \left(\sum_j \frac{\partial h}{\partial z_j}\xi_j\right)\left(\sum_k \frac{\partial \rho}{\partial \bar{z}_k}\bar{\xi}_k\right).$$

Now (11) implies that $\sum_j(\partial\rho/\partial z_j)\xi_j = 0$ on $\partial\Omega$. Also $\overline{\partial\rho/\partial z_k} = \partial\rho/\partial\bar{z}_k$, whence $\sum_k(\partial\rho/\partial\bar{z}_k)\bar{\xi}_k = 0$ on $\partial\Omega$. Since $\rho = 0$ on $\partial\Omega$ and $h > 0$ there, (2) implies that

(12) $$\text{On } \partial\Omega, \sum_{j,k}\frac{\partial^2 R}{\partial z_j \partial \bar{z}_k}\xi_j\bar{\xi}_k \geq 0 \text{ if } \sum_j \frac{\partial R}{\partial z_j}\xi_j = 0.$$

Now choose a function h as above with $h = 1/|\text{grad } \rho|$ in a neighborhood of $\partial\Omega$. Then $R = h \cdot \rho = \rho/|\text{grad } \rho|$ there, whence $|\text{grad } R| = 1$ on $\partial\Omega$. Also Ω is defined by $R < 0$ and (12) holds.

The upshot is that we may without loss of generality suppose that $|\text{grad } \rho| = 1$ on $\partial\Omega$. It then holds that grad ρ is the outer unit normal to $\partial\Omega$ at each point of $\partial\Omega$. The divergence theorem now gives for every smooth function v on $\bar{\Omega}$,

(13) $$\int_\Omega \frac{\partial v}{\partial x_j}dV = \int_{\partial\Omega} v\frac{\partial \rho}{\partial x_j}dS$$

for all real coordinates $x_1, \ldots, x_{2n}$ in $\mathbb{C}^n$. Hence for $1 \leq j \leq n$,

(14) $$\int_\Omega \frac{\partial v}{\partial \bar{z}_j}dV = \int_{\partial\Omega} v\frac{\partial \rho}{\partial \bar{z}_j}dS.$$

Now fix $f = \sum_1^n f_j d\bar{z}_j \in C^1_{0,1}(\bar{\Omega})$, and fix $u \in C^1(\bar{\Omega})$. Then with $(\ ,\ )_j$ denoting the inner product in H_j as defined above,

$$(Tu, f)_2 = \left(\sum_j \frac{\partial u}{\partial \bar{z}_j}d\bar{z}_j, \sum_j f_j d\bar{z}_j\right)_2$$
$$= \int_\Omega\left(\sum_j \frac{\partial u}{\partial \bar{z}_j}\bar{f}_j\right)e^{-\phi}dV.$$

Fix j. Then

$$\begin{aligned}\int_\Omega \frac{\partial u}{\partial \bar{z}_j} \bar{f}_j e^{-\phi} dV \\ &= \int_\Omega \frac{\partial}{\partial \bar{z}_j}(u\bar{f}_j e^{-\phi})dV - \int_\Omega u \frac{\partial}{\partial \bar{z}_j}(\bar{f}_j e^{-\phi})dV \\ &= \int_{\partial\Omega} u\bar{f}_j e^{-\phi}\frac{\partial \rho}{\partial \bar{z}_j} dS - \int_\Omega u \frac{\partial}{\partial \bar{z}_j}(\bar{f}_j e^{-\phi})dV,\end{aligned}$$

where we have used (14). Hence we have

$$\begin{aligned}(Tu, f)_2 = &- \int_\Omega u \left[\sum_j \frac{\partial}{\partial \bar{z}_j}(\bar{f}_j e^{-\phi})\right] dV \\ &+ \int_{\partial\Omega} u \left(\sum_j \bar{f}_j \frac{\partial \rho}{\partial \bar{z}_j}\right) e^{-\phi} dS.\end{aligned}$$

Now if $f \in \mathcal{D}_{T^*}$, it follows that we also have

$$(Tu, f)_2 = \int_\Omega u\overline{T^* f} e^{-\phi} dV.$$

Since the last two equations hold for all u in $C^1(\bar{\Omega})$, we conclude that

$$\sum_j \bar{f}_j \frac{\partial \rho}{\partial \bar{z}_j} = 0 \text{ on } \partial\Omega, \tag{15}$$

which yields (9), and that

$$\begin{aligned}\overline{T^* f} e^{-\phi} &= -\sum \frac{\partial}{\partial \bar{z}_j}(\bar{f}_j e^{-\phi}) \\ &= \overline{-\sum_j \frac{\partial}{\partial z_j}(f_j e^{-\phi})}, \text{ whence (10).}\end{aligned}$$

□

Define an operator δ_j by

$$\delta_j w = e^\phi \frac{\partial}{\partial z_j}(we^{-\phi}).$$

Fix $f \in C^1_{0,1}(\bar{\Omega}) \cap \mathcal{D}_{T^*}$. By (10), $T^* f = -\sum_j \delta_j f_j$, and so

$$||T^* f||_1^2 = \sum_{j,k} \int_\Omega \delta_j f_j \cdot \overline{\delta_k f_k} e^{-\phi} dV. \tag{16}$$

Now fix $A, B \in C^1(\bar{\Omega})$. Applying (14) with $v = A\bar{B}e^{-\phi}$ and $j = \nu$ gives

$$\int_\Omega \frac{\partial}{\partial \bar{z}_\nu}(A\bar{B}e^{-\phi})dV = \int_{\partial\Omega} A\bar{B}e^{-\phi}\frac{\partial \rho}{\partial \bar{z}_\nu} dS.$$

Hence

$$\int_\Omega \frac{\partial A}{\partial \bar z_\nu}\,\bar B e^{-\phi} dV = -\int_\Omega A \frac{\partial}{\partial \bar z_\nu}(\bar B e^{-\phi}) dV + \int_{\partial\Omega} A\bar B e^{-\phi}\frac{\partial \rho}{\partial \bar z_\nu} dS$$
$$= -\int_\Omega A\overline{\delta_\nu B} e^{-\phi} dV + \int_{\partial\Omega} A\bar B \frac{\partial\rho}{\partial \bar z_\nu} e^{-\phi} dS.$$

Writing $\int_\Omega(\,)$ for $f(\,)e^{-\phi}dV$ and similarly for $\partial\Omega$, we thus have

$$\int_\Omega \frac{\partial A}{\partial \bar z_\nu}\overline{B} = -\int_\Omega A\overline{\delta_\nu B} + \int_{\partial\Omega} A\bar B\frac{\partial\rho}{\partial \bar z_\nu}. \tag{17}$$

Putting $A = \delta_k w$, $B = v$, and $\nu = j$ in (17) gives

$$\int_\Omega \frac{\partial}{\partial \bar z_j}(\delta_k w)\cdot \bar v = -\int_\Omega \delta_k w\cdot\overline{\delta_j v} + \int_{\partial\Omega}\delta_k w\cdot \bar v\frac{\partial\rho}{\partial \bar z_j}. \tag{18}$$

Direct computation gives for all u

$$\left(\delta_k\frac{\partial}{\partial \bar z_j} - \frac{\partial}{\partial \bar z_j}\delta_k\right)(u) = \frac{\partial^2\phi}{\partial \bar z_j\partial z_k}\cdot u,$$

so

$$-\frac{\partial}{\partial \bar z_j}(\delta_k w) = \frac{\partial^2\phi}{\partial \bar z_j\partial z_k} w - \delta_k\left(\frac{\partial w}{\partial \bar z_j}\right).$$

Hence

$$\int_\Omega -\frac{\partial}{\partial \bar z_j}(\delta_k w)\cdot\bar v = \int_\Omega \frac{\partial^2\phi}{\partial \bar z_j\partial z_k} w\bar v - \int_\Omega \delta_k\left(\frac{\partial w}{\partial \bar z_j}\right)\bar v. \tag{19}$$

Putting $A = v$, $B = \partial w/\partial \bar z_j$, and $\nu = k$ in (17), we get

$$\int_\Omega \frac{\partial v}{\partial \bar z_k}\overline{\frac{\partial w}{\partial \bar z_j}} = -\int_\Omega v\overline{\delta_k\left(\frac{\partial w}{\partial \bar z_j}\right)} + \int_{\partial\Omega} v\overline{\frac{\partial w}{\partial \bar z_j}}\frac{\partial\rho}{\partial \bar z_k}, \tag{20}$$

which combined with the complex conjugate of (19) gives

$$\overline{\int_\Omega -\frac{\partial}{\partial \bar z_j}(\delta_k w)\bar v} = \int_\Omega \frac{\partial^2\phi}{\partial z_j\partial \bar z_k}\bar w v - \int_\Omega v\overline{\delta_k\left(\frac{\partial w}{\partial \bar z_j}\right)}$$
$$= \int_\Omega \frac{\partial^2\phi}{\partial z_j\partial \bar z_k}\bar w v - \int_{\partial\Omega} v\overline{\frac{\partial w}{\partial \bar z_j}}\frac{\partial\rho}{\partial \bar z_k} + \int_\Omega \frac{\partial v}{\partial \bar z_k}\overline{\frac{\partial w}{\partial \bar z_j}}. \tag{21}$$

Combining (21) with the complex conjugate of (18) gives

$$\int_\Omega \delta_j v\cdot\overline{\delta_k w} = \int_\Omega \frac{\partial^2\phi}{\partial z_j\partial \bar z_k}\bar w v + \int_\Omega \frac{\partial v}{\partial \bar z_k}\overline{\frac{\partial w}{\partial \bar z_j}}$$
$$- \int_{\partial\Omega} v\overline{\frac{\partial w}{\partial \bar z_j}}\frac{\partial\rho}{\partial \bar z_k} + \int_{\partial\Omega}\overline{\delta_k w}\cdot v\frac{\partial\rho}{\partial z_j}. \tag{22}$$

By (16),

$$||T^* f||_1^2 = \sum_{j,k} \int_\Omega \delta_j f_j \overline{\delta_k f_k},$$

so

$$(23) \qquad ||T^* f||_1^2 = \int_\Omega \sum_{j,k} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} f_j \bar{f}_k + \int_\Omega \sum_{j,k} \frac{\partial f_j}{\partial \bar{z}_k} \overline{\frac{\partial f_k}{\partial \bar{z}_j}}$$
$$- \int_{\partial\Omega} \sum_{j,k} f_j \overline{\frac{\partial f_k}{\partial \bar{z}_j}} \frac{\partial \rho}{\partial \bar{z}_k} + \int_{\partial\Omega} \sum_{j,k} \overline{\delta_k f_k} \cdot f_j \frac{\partial \rho}{\partial z_j}.$$

Assertion.

$$- \sum_{j,k} f_j \overline{\frac{\partial f_k}{\partial \bar{z}_j}} \frac{\partial \rho}{\partial \bar{z}_k} = \sum_{j,k} f_j \bar{f}_k \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \text{ on } \partial\Omega.$$

For, by (9),

$$\sum_k f_k \frac{\partial \rho}{\partial z_k} = 0 \text{ on } \partial\Omega.$$

Hence the gradient of the function $\sum_k f_k(\partial\rho/\partial z_k)$ is a scalar multiple of grad ρ. Hence $\exists$ function λ on $\partial\Omega$ with

$$\frac{\partial}{\partial \bar{z}_j} \left(\sum_k f_k \frac{\partial \rho}{\partial z_k} \right) = \lambda \frac{\partial \rho}{\partial \bar{z}_k}, \qquad j = 1, 2, \ldots, n,$$

or

$$\sum_k \frac{\partial f_k}{\partial \bar{z}_j} \frac{\partial \rho}{\partial z_k} + \sum_k f_k \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_k} = \lambda \frac{\partial \rho}{\partial \bar{z}_j}.$$

Multiplying by $\bar{f}_j$ and summing over j gives

$$\sum_{j,k} \bar{f}_j \frac{\partial f_k}{\partial \bar{z}_j} \frac{\partial \rho}{\partial z_k} + \sum_{j,k} \bar{f}_j f_k \frac{\partial^2 \rho}{\partial \bar{z}_j \partial z_k} = \lambda \sum_j \bar{f}_j \frac{\partial \rho}{\partial \bar{z}_j} = \lambda \overline{\sum_j f_j \frac{\partial \rho}{\partial z_j}} = 0.$$

Complex conjugate now gives the assertion. The last term on the right in (23)

$$= \int_{\partial\Omega} \left(\sum_k \overline{\delta_k f_k} \right) \left(\sum_j f_j \frac{\partial \rho}{\partial z_j} \right) = 0, \quad \text{by (9).}$$

Equation (23) and the assertion now yield

Lemma 16.5. *Fix $f \in \mathcal{D}_{T^*} \cap C^1_{0,1}(\bar{\Omega})$. Then*

(24)

$$||T^* f||_1^2 = \int_\Omega \sum_{j,k} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k} f_j \bar{f}_k + \int_\Omega \sum_{j,k} \frac{\partial f_j}{\partial \bar{z}_k} \overline{\frac{\partial f_k}{\partial \bar{z}_j}} + \int_{\partial\Omega} \sum_{j,k} f_j \bar{f}_k \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}.$$

Lemma 16.6. *Fix $f \in \mathcal{D}_S \cap C^1_{0,1}(\bar\Omega)$. Then*

$$||Sf||_3^2 = \int_\Omega \sum_{j,k} \left|\frac{\partial f_k}{\partial \bar z_j}\right|^2 - \int_\Omega \sum_{j,k} \frac{\partial f_j}{\partial \bar z_k}\overline{\frac{\partial f_k}{\partial \bar z_j}}. \tag{25}$$

Proof. Since $f \in C^1_{0,1}(\bar\Omega)$. Then

$$\begin{aligned} Sf = \bar\partial f &= \sum_\alpha \left(\sum_\beta \frac{\partial f_\alpha}{\partial \bar z_\beta} d\bar z_\beta\right) \wedge d\bar z_\alpha \\ &= \sum_{\alpha<\beta}\left(\frac{\partial f_\beta}{\partial \bar z_\alpha} - \frac{\partial f_\alpha}{\partial \bar z_\beta}\right) d\bar z_\alpha \wedge d\bar z_\beta. \end{aligned}$$

Hence

$$\begin{aligned} ||Sf||_3^2 &= \int_\Omega \sum_{\alpha<\beta}\left(\frac{\partial f_\beta}{\partial \bar z_\alpha} - \frac{\partial f_\alpha}{\partial \bar z_\beta}\right)\left(\overline{\frac{\partial f_\beta}{\partial \bar z_\alpha}} - \overline{\frac{\partial f_\alpha}{\partial \bar z_\beta}}\right) \\ &= \int_\Omega \sum_{\alpha<\beta}\left|\frac{\partial f_\beta}{\partial \bar z_\alpha}\right|^2 + \int_\Omega \sum_{\alpha<\beta}\left|\frac{\partial f_\alpha}{\partial \bar z_\beta}\right|^2 \\ &\quad - \int_\Omega \sum_{\alpha<\beta} \frac{\partial f_\beta}{\partial \bar z_\alpha}\overline{\frac{\partial f_\alpha}{\partial \bar z_\beta}} - \int_\Omega \sum_{\alpha<\beta} \frac{\partial f_\alpha}{\partial \bar z_\beta}\overline{\frac{\partial f_\beta}{\partial \bar z_\alpha}}, \end{aligned}$$

Which coincides with (25) □

Proof of Theorem 16.3. Adding equations (24) and (25) gives (7). □

Note. The proof of Theorem 16.1 is now complete.

In the rest of this section we shall establish some regularity properties of solutions of the equation $\bar\partial u = f$, given information on f.

Lemma 16.7. *Put $B = \{z \in \mathbb{C}^n \,||z| < 1\}$. There exists a constant K such that for $w \in C^\infty(\mathbb{C}^n)$,*

$$|w(0)| \le K\left\{||w||_{L^2(B)} + \sup_B\left(\max_j\left|\frac{\partial w}{\partial \bar z_j}\right|\right)\right\}. \tag{26}$$

Proof. It is a fact form classical potential theory that if $f \in C_0^\infty(\mathbb{R}^N)$, then

$$f(y) = C\int_{\mathbb{R}^N} \Delta F \frac{dx}{|x-y|^{N-2}}, \tag{27}$$

where C is a constant depending on N and dx is Lebesgue measure on $\mathbb{R}^N$.

Now let $\chi \in C^\infty(\mathbb{C}^n)$, supp $\chi \subset B$, and $\chi = 1$ in $|z| < \frac{1}{2}$. Then by (27) with $y = 0$ and $f = \chi w$,

$$w(0) = (\chi w)(0) = \int_{\mathbb{C}^n} \Delta(\chi w)E(x)dx,$$

where we put $E(x) = C/|x|^{2n-2}$. Thus

$$w(0) = I_1 + 2I_2 + I_3,$$

where

$$I_1 = \int \Delta\chi \cdot wE\,dx,$$
$$= I_3 \int \Delta w \cdot \chi E\,dx,$$

and

$$I_2 = \int (\text{grad}\,\chi, \text{grad}\,w)E\,dx.$$

With x_j the real coordinates in $\mathbb{C}^n$, we have

$$\int \chi_{x_i} w_{x_i} E\,dx = \int w_{x_i}(\chi_{x_i}E)dx = -\int w(\chi_{x_i}E)_{x_i}dx,$$

so $I_2 = -\int w \sum_i (\chi_{x_i}E)_{x_i}\,dx$.

Since χ_{x_i} and $\Delta\chi$ vanish in a neighborhood of 0 and supp $\chi \subset B$, we have, with K a constant,

$$|I_1| \leq K||w||_{L^2(B)}, \qquad |I_2| \leq K||w||_{L_2(B)}.$$

Also,

$$I_3 = \int 4\left(\sum_j \frac{\partial^2 w}{\partial z_j \partial \bar{z}_j}\right)\chi E\,dx$$
$$= 4\sum_j \int \frac{\partial}{\partial z_j}\left(\frac{\partial w}{\partial \bar{z}_j}\right)\chi E\,dx = -4\sum_j \int \frac{\partial w}{\partial \bar{z}_j}\frac{\partial}{\partial z_j}(\chi E)\,dx.$$

Since $\partial E/\partial x_j \in L^1$ locally, we have

$$|I_3| \leq K \sup_B \left(\max_j \left|\frac{\partial w}{\partial \bar{z}_j}\right|\right).$$

Equation (26) follows. □

Choose $\chi \in C^\infty(\mathbb{C}^n)$, $\chi \geq 0$, $\chi(6) = 0$ for $|y| > 1$, and $\int \chi(y)dy = 1$, where we write dy for Lebesgue measure on $\mathbb{C}^n$. Put $\chi_\varepsilon(y) = (1/\varepsilon^{2n})\chi(y/\varepsilon)$. then for every $\varepsilon > 0$,

$$\chi_\varepsilon \in C^\infty(\mathbb{C}^n), \qquad \chi_\varepsilon(y) = 0 \text{ for } |y| > \varepsilon,$$

$\int \chi_\varepsilon(y)dy = 1$.

Let now $u \in L^2(\mathbb{C}^n)$ and put

$$u_\varepsilon(x) = \int u(x-y)\chi_\varepsilon(y)dy.$$

Note that this integral converges absolutely for all x. We assert that

(28) $u_\varepsilon \in C^\infty(\mathbb{C}^n)$.

(29) $u_\varepsilon \to u$ in $L^2(\mathbb{C}^n)$, as $\varepsilon \to 0$.

(30) If u is continuous in a neighborhood of a closed ball, then $u_\varepsilon \to u$ uniformly on the ball.

The proofs of (28), (29), and (30) are left to the reader.

Lemma 16.8. *Let $B = \{z \in \mathbb{C}^n \big| |z| < 1\}$. Let $u \in L^2(B)$. Assume that for each j, $\partial u/\partial \bar z_j$, defined as distribution on B, is continuous. (Recall Definition 16.1.) Then u is continuous and (26) holds with $w = u$.*

Proof. Fix $x \in \mathbb{C}^n$ and $r > 0$ and put $B(x, r) = \{z \in \mathbb{C}^n \big| |z - x| < r\}$. A linear change of variable converts (26) into

$$|w(x)| \le K \left\{ r^{-n} ||w||_{L^2(B(x,r))} + r \sup_{B(x,r)} \left(\max_j \left| \frac{\partial w}{\partial \bar z_j} \right| \right) \right\}. \tag{31}$$

Extend u to all of $\mathbb{C}^n$ by putting $u = 0$ outside B. Then $u \in L^2(\mathbb{C}^n)$. For each $\rho > 0$, put $B_\rho = \{z \big| |z| < \rho\}$. Fix $R < 1$ and fix $r < 1 - R$. For each $x \in B_R$, then, $B(x, r) \subset B_{R+r} = B'$.

Fix $x \in B_R$. If $\varepsilon, \varepsilon' > 0$, $u_\varepsilon - u_{\varepsilon'} \in C^\infty(\mathbb{C}^n)$. Equation (31) together with $B(x, r) \subset B'$ gives

$$\begin{aligned} &|u_\varepsilon(x) - u_{\varepsilon'}(x)| \\ &\quad \le K \left\{ r^{-n} ||u_\varepsilon - u_{\varepsilon'}||_{L^2(B'')} + r \sup_{B'} \left(\max_j \left| \frac{\partial u_\varepsilon}{\partial \bar z_j} - \frac{\partial u_{\varepsilon'}}{\partial \bar z_j} \right| \right) \right\}. \end{aligned}$$

Now, by (29), $||u_\varepsilon - u_{\varepsilon'}||_{L_2(B')} \to 0$ as $\varepsilon, \varepsilon' \to 0$. Also, it is easy to see that $\partial u_\varepsilon/\partial \bar z_j - \partial u_{\varepsilon'}/\partial \bar z_j \to 0$ uniformly on B' as $\varepsilon, \varepsilon' \to 0$. Hence $u_\varepsilon(x) - u_{\varepsilon'}(x) \to 0$ uniformly for $x \in B_R$. Hence $U = \lim_{\varepsilon \to 0} u_\varepsilon$ is continuous in B_R. Also, by (29), $u_\varepsilon \to u$ in $L^2(B)$. Hence $U = u$ and so u is continuous in B_R. It follows that u is continuous in B, as claimed.

Fix $\varepsilon > 0$ and $\rho < 1$. Then, by (31),

$$|u_\varepsilon(0)| \le K \left\{ \rho^{-n} ||u_\varepsilon||_{L^2(B_\rho)} + \rho \sup_{B_\rho} \left(\max_j | \frac{\partial u_\varepsilon}{\partial \bar z_j} | \right) \right\}.$$

As $\varepsilon \to 0$, $u_\varepsilon(0) \to u(0)$, $||u_\varepsilon||_{L_2(B_\rho)} \to ||u||_{L_2(B_\rho)}$, and $\partial u_\varepsilon/\partial \bar z_j \to \partial u/\partial \bar z_j$ uniformly on B_ρ for each j. Hence

$$|u(0)| \le K \left\{ \rho^{-n} ||u||_{L^2(B_\rho)} + \rho \sup_{B_\rho} \left(\max_j | \frac{\partial u}{\partial \bar z_j} | \right) \right\}.$$

Letting $\rho \to 1$, we get that (26) holds with $w = u$. □

Lemma 16.9. *Let Ω be a bounded domain in $\mathbb{C}^n$ and $u \in L^2(\Omega)$. Assume that for all j,*

$$\frac{\partial u}{\partial \bar{z}_j} = 0 \text{ as a distribution on } \Omega. \tag{32}$$

Then $u \in H(\Omega)$.

PROOF. Define $u = 0$ outside Ω. Then $u \in L^2(\mathbb{C}^n)$. By a change of variable, we get

$$u_\varepsilon(z) = \inf u(\zeta)\chi_\varepsilon(z-\zeta)d\zeta.$$

Fix j. Note that $(\partial\{\chi_\varepsilon(z-\zeta)\}/\partial\bar{z}_j) = -(\partial\{\chi_\varepsilon(z-\zeta)\}/\partial\bar{\zeta}_j)$. Hence

$$\frac{\partial u_\varepsilon}{\partial \bar{z}_j}(z) = \int u(\zeta)\frac{\partial}{\partial \bar{z}_j}(\chi_\varepsilon(1-\zeta))d\zeta = -\int u(\zeta)\frac{\partial}{\partial \bar{\zeta}_j}(\chi_\varepsilon(z-\zeta))d\zeta.$$

Fix $z \in \Omega$ and choose $\varepsilon < \operatorname{dist}(z, \partial\Omega)$. Put $g(\zeta) = \chi_\varepsilon(z-\zeta)$. Then supp g is a compact subset of Ω. By (32),

$$\int u(\zeta)\frac{\partial g}{\partial \bar{\zeta}_j}(\zeta)d\zeta = 0.$$

Thus $\partial u_\varepsilon(z)/\partial\bar{z}_j = 0$. Hence $u_\varepsilon \in H(\Omega)$.

Fix a closed ball $B' \subset \Omega$. By (32), $\partial u/\partial\bar{z}_j$ is continuous in a neighborhood of B' and so, by (30), $u_\varepsilon \to u$ uniformly in B' as $\varepsilon \to 0$. Hence $u \in H(\mathring{B}')$. So $u \in H(\Omega)$. □

NOTES

The fundamental result of this section, Theorem 16.1, is due to L. Hörmander. It is proved in considerably greater generality in Hörmander's paper, L^2 estimates and existence theorems for the $\bar{\partial}$-operator. We have followed the proof in that paper, restricting ourselves to (0, 1)-forms. The method of proving existence theorems for the $\bar{\partial}$-operator by means of L^2 estimates was developed by C. B. Morrey, The analytic embedding of abstract real analytic manifolds, *Ann. Math.* (2), **68** (1958), and J. J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds, I and II, *Ann. Math.* (2), **78** (1963) and *Ann. Math.* (2), **79** (1964). These methods have proved to be powerful tools in many questions concerning analytic functions of several complex variables. For such applications the reader may consult, e.g., Hörmander's book *An Introduction to Complex Analysis in Several Variables* [Hö2, Chaps. IV and V].

In section 17 we shall apply Theorem 16.1 to a certain approximation problem.

17

Manifolds Without Complex Tangents

Let X be a compact set in $\mathbb{C}^n$ which lies on a smooth k-dimensional (real) submanifold $\sum$ of $\mathbb{C}^n$. Assume that X is polynomially convex. Under what conditions on $\sum$ can we conclude that $P(X) = C(X)$?

If $\sum$ is a complex-analytic submanifold of $\mathbb{C}^n$, it does not have this property. On the other hand, the real subspace $\sum_R$ of $\mathbb{C}^n$ does have this property. What feature of the geometry of $\sum$ is involved?

Now fix a k-dimensional smooth submanifold $\sum$ of an open set in $\mathbb{C}^n$, and consider a point $x \in \sum$. Denote by T_x the tangent space to $\sum$ at x, viewed as a real-linear subspace of $\mathbb{C}^n$.

Definition 17.1. A *complex tangent* to $\sum$ at x is a complex line, i.e., a complex-linear subspace of $\mathbb{C}^n$ of complex dimension 1, contained in T_x.

Note that if $\sum$ is complex-analytic, then it has one or more complex tangents at every point. whereas $\sum_R$ has no complex tangent whatever.

Definition 17.2. Let Ω be an open set in $\mathbb{C}^n$ and let $\sum$ be a closed subset of Ω. $\sum$ is called a *k-dimensional submanifold of Ω of class e* if for each x_0 in $\sum$ we can find a neighborhood U of x_0 in $\mathbb{C}^n$ with the following property: There exist real-valued functions $\rho_1, \rho_2, \ldots, \rho_{2n-k}$ in $C^e(U)$ such that

$$\sum \cap U = \{x \in U | \rho_j(x) = 0,\ j = 1, 2, \ldots, 2n - k\},$$

and such that the matrix $(\partial\rho_j/\partial x_v)$, where $x_1, x_2, \ldots, x_{2n}$ are the real coordinates in $\mathbb{C}^n$, has rank $2n - k$.

EXERCISE 17.1. Let $\sum, \rho_1, \ldots, \rho_{2n-k}$, be as above and fix $x^0 \in \sum$. If there exists a tangent vector ξ to $\sum$ at x^0 of the form

$$\xi = \sum_{j=1}^{n} c_j \frac{\partial}{\partial \bar{z}_j}$$

such that $\xi(\rho_v) = 0$, all v, then $\sum$ has a complex tangent at x^0.

Theorem 17.1. *Let $\sum$ be a k-dimensional sufficiently smooth submanifold of an open set in $\mathbb{C}^n$. Assume that $\sum$ has no complex tangents.*

Let X be a compact polynomially convex subset of $\sum$. Then $P(X) = C(X)$.

Note 1. "Sufficiently smooth" will mean that $\sum$ is of class e with $e > (k/2) + 1$. It is possible that class 1 would be enough to give the conclusion.

Note 2. After proving Theorem 17.1, we shall use it in Theorem 17.5 to solve a certain perturbation problem.

Sketch of Proof. To show that $P(X) = C(X)$ we need only show that $P(X)$ contains the restriction to every X of every $u \in C^\infty(\mathbb{C}^n)$, since such functions are dense in $C(X)$.

Fix $u \in C^\infty(\mathbb{C}^n)$. By the Oka-Weil theorem it suffices to approximate u uniformly on X by functions defined and holomorphic in some neighborhood of X in $\mathbb{C}^n$. To this end, we shall do the following:

Step 1. Construct for each $\varepsilon > 0$ a certain neighborhood ω_ε of X in $\mathbb{C}^n$ to which Theorem 16.1 is applicable.

Step 2. Find an extension U_ε of $u|_X$ to ω_ε such that $\bar\partial U_\varepsilon$ is "small" in ω_ε.

Step 3. Using the results of Section 16, find a function V_ε in ω_ε such that $\bar\partial V_\varepsilon = \bar\partial U_\varepsilon$ in ω_ε and $\sup_X |V_\varepsilon| \to 0$ as $\varepsilon \to 0$,

Once step 3 is done, we write

$$U_\varepsilon = (U_\varepsilon - V_\varepsilon) + V_\varepsilon \text{ in } \omega_\varepsilon.$$

Then $U_\varepsilon - V_\varepsilon$ is holomorphic in ω_ε, since $\bar\partial(U_\varepsilon - V_\varepsilon) = 0$ by step 3. Since $\sup_X |V_\varepsilon| \to 0$, this holomorphic function approximates $u = U_\varepsilon$ as closely as we please on X.

Definition 17.3. Let Ω be an open set in $\mathbb{C}^n$ and fix $F \in C^2(\Omega)$. F is *plurisubharmonic* (p.s.) in Ω if

$$\sum_{j,k=1}^{n} \frac{\partial^2 F}{\partial z_j \partial \bar z_k}(z)\xi_j\bar\xi_k \geq 0 \tag{1}$$

if $z \in \Omega$ and $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$.

F is *strongly* p.s. in Ω if the inequality in (1) is strict, except when $(\xi_1, \ldots, \xi_n) = 0$.

Lemma 17.2. *Let $\sum$ be a submanifold of an open set in $\mathbb{C}^n$ of class 2 such that $\sum$ has no complex tangents. Let d be the distance function to $\sum$; i.e., if $x \in \mathbb{C}^n$, $d(x)$ is the distance from x to $\sum$. Then $\exists$ a neighborhood ω of $\sum$ such that $d^2 \in C^2(\omega)$ and d^2 is strongly p.s. in ω.*

EXERCISE 17.2. Prove the smoothness assertion; i.e., show that d^2 is in C^2 in some neighborhood of $\sum$.

PROOF OF LEMMA 17.2. Let U be a neighborhood of $\sum$ such that $d^2 \in C^2(U)$.

Fix $z_0 \in \sum$. We assert that

$$\sum_{j,k=1}^{n} \frac{\partial^2(d^2)}{\partial z_j \partial \bar{z}_k}(z_0)\xi_j\bar{\xi}_k > 0 \tag{2}$$

for all $\xi = (\xi_1, \ldots, \xi_n)$ with $\xi \neq 0$.

Without loss of generality $z_0 = 0$. Let T be the tangent space to $\sum$ at 0 and put $d(z, T) =$ distance from z to T.

*EXERCISE 17.3.

$$d^2(z) = d^2(z, T) + o(|z|^2). \tag{3}$$

Also

$$d^2(z, T) = H(z) + \operatorname{Re} A(z), \tag{4}$$

where $H(z) = \sum_{j,k=1}^{n} h_{jk} z_j \bar{z}_k$ is hermitean-symmetric and A is a homogeneous quadratic polynomial in z.

Equations (3) and (4) imply that

$$\sum_{j,k=1}^{n} \frac{\partial^2(d^2)}{\partial z_j \partial \bar{z}_k}(0) z_j \bar{z}_k = H(z). \tag{5}$$

Now

$$d^2(z, T) + d^2(iz, T) = 2H(z).$$

If $z \neq 0$, either z of $iz \notin T$, since by hypothesis T contains no complex line. Hence $H(z) > 0$. Because of (5), this shows that (2) holds.

It follows by continuity from (2) that

$$\sum_{j,k=1}^{n} \frac{\partial^2(d^2)}{\partial z_j \partial \bar{z}_k}(z)\xi_j\bar{\xi}_k > 0$$

for all z in some neighborhood of $\sum$ and $\xi \notin 0$. □

From now on until the end of the proof of Theorem 17.1 let $\sum$ and X be as in that theorem and let d be as in Lemma 17.2.

Lemma 17.3. *There exists an open set ω_ε in $\mathbb{C}^n$ containing X such that ω_ε is bounded and*

(6) *If $z \in \omega_\varepsilon$, then $d(z) < \varepsilon$.*

(7) *If $z_0 \in X$ and $|z - z_0| < \varepsilon/2$, then $z \in \omega_\varepsilon$.*

(8) $\exists$ *a function* u_ε *in* C^∞ *in some neighborhood of* $\bar{\omega}_\varepsilon$ *such that* ω_ε *is defined by*

$$u_\varepsilon(z) < 0.$$

(9) $u_\varepsilon = 0$ *on* $\partial\omega_\varepsilon$ *and grad* $u_\varepsilon \neq 0$ *on* $\partial\omega_\varepsilon$.

(10) u_ε *is p.s. in a neighborhood of* $\bar{\omega}_\varepsilon$.

PROOF. Choose ω by Lemma 17.2 so that d^2 is strongly p.s. in ω. Next choose $\beta \in C_0^\infty(\omega)$ with $\beta = 1$ in a neighborhood of X and $0 \leq \beta \leq 1$. Let Ω be an open set with compact closure such that

$$\operatorname{supp} \beta \subset \Omega \subset \bar{\Omega} \subset \omega.$$

Since d^2 is strongly p.s. in ω, we can choose $\varepsilon > 0$ such that

$$\phi = d^2 - \varepsilon^2\beta$$

is p.s. in Ω. Further, choose ε so small that $\beta(z) = 1$ for each z whose distance from $X < \varepsilon$. Next, choose an open set Ω_1 with

$$\operatorname{supp} \beta \subset \Omega_1 \subset \bar{\Omega}_1 \subset \Omega.$$

Assertion. $\exists \in C^\infty(\mathbb{C}^n)$ such that u is p.s. in Ω_1 and

$$|u - \phi_1 < \frac{\varepsilon^2}{4} \text{ on } \Omega_1. \tag{11}$$

We proceed as in the last part of Section 16. Choose $\chi \in C^\infty(\mathbb{C}^N)$, $\chi \geq 0$, $\chi(y) = 0$ for $|y| > 1$ and $\int \chi(y)dy = 1$. Put $\chi_\delta(y) = (1/\delta^{2n})\chi(y/\delta)$ and put

$$\phi_\delta(x) = \int \phi(x - y)\chi_\delta(y)dy,$$

where we have defined $\phi = 0$ outside Ω.

Then, as in Section 16, if δ is small,

$$\phi_\delta \in C^\infty(\mathbb{C}^n). \tag{12}$$

$$\phi_\delta \to \phi \text{ uniformly on } \Omega_1 \text{ as } \delta \to 0. \tag{13}$$

Also for each $(\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$, $z \in \Omega_1$:

$$\sum_{j,k} \frac{\partial^2\phi_\delta}{\partial z_j \partial \bar{z}_k}(z)\xi_j\bar{\xi}_k = \int \left\{ \sum_{j,k} \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_k}(z - y)\xi_j\bar{\xi}_k \right\} \chi_\delta(y)dy \geq 0,$$

since ϕ is p.s. in Ω. Hence

$$\phi_\delta \text{is p.s. in } \Omega_1. \tag{14}$$

Choose δ such that $|\phi - \phi_\delta| < \varepsilon^2/4$ on Ω_1 and put $u = \phi_\omega$. Thus the assertion holds.

Since $u \in C^\infty(\mathbb{C}^n)$, a well-known theorem yields that the image under u of the set grad $u = 0$ has measure 0 on $\mathbb{R}$. Hence every interval on $\mathbb{R}$ contains a point t such that the level set $u = t$ fails to meet the set grad $u = 0$. Choose such a t with

$$-\tfrac{1}{2}\varepsilon^2 < t < -\tfrac{1}{4}\varepsilon^2.$$

Define

$$\omega_\varepsilon = \{x \in \Omega_1 | u(x) < t\}.$$

We claim that ω_ε has the required properties. Put

$$u_\varepsilon = u - t.$$

Then $\omega_\varepsilon = \{x \in \Omega_1 | u_\varepsilon < 0\}$. It is easily verified that $\omega_\varepsilon \subset \operatorname{supp}\beta$. It follows that $u_\varepsilon = 0$ on $\partial\omega_\varepsilon$.

Since $u = t$ on ω_ε, it follows by choice of t that grad u, and hence grad u_ε, $\neq 0$ on $\partial\omega_\varepsilon$. Thus (8) and (9) hold and (10) holds since u is p.s. in Ω_1.

Equations (6) and (7) are verified directly, using (11) and the fact that $-\varepsilon^2/2 < t < -\varepsilon^2/4$.

Thus the lemma is established. This completes step 1.

Lemma 17.4. *Fix a compact set K on $\sum$. Let u be a function of class C^e defined on $\sum$. Then $\exists$ a function U of class C^1 in $\mathbb{C}^n$ with*
(a) *$U \equiv u$ on K.*
(b) *$\exists$ constant C with*

$$\left|\frac{\partial U}{\partial \bar{z}_j}(z)\right| \leq C \cdot d(z)^{e-1}, \qquad \textit{all } z,\ j = 1, \ldots, n.$$

Proof. We first perform the extension locally.

Fix $x_0 \in \sum$. Choose an open set Ω in $\mathbb{C}^n$ such that $x_0 \in \Omega$, and choose real functions ρ_j such that

$$\sum \cap \Omega = \{x \in \Omega | \rho_1(x) = \cdots = \rho_m(x) = 0\},$$

where each ρ_j is of class C^e in Ω and such that u has an extension to $C^e(\Omega)$, again denoted u.

We assert that $\exists$ a neighborhood ω_0 of x_0 and $\exists$ integers $v_1, v_2, \ldots, v_n$ such that the vectors

$$\left(\frac{\partial \rho_{v_j}}{\partial \bar{z}_1}, \ldots, \frac{\partial \rho_{v_j}}{\partial \bar{z}_n}\right)_x, \qquad j = 1, \ldots, n$$

form a basis for $\mathbb{C}^n$ for each $x \in \omega_0$.

Put

$$\xi_v = \left(\frac{\partial \rho_v}{\partial \bar{z}_1}, \ldots, \frac{\partial \rho_v}{\partial \bar{z}_n}\right)_{x_0}, \qquad v = 1, \ldots, m.$$

Suppose that $\xi_1, \ldots, \xi_m$ fail to span $\mathbb{C}^n$. Then $\exists c = (c_1, \ldots, c_n) \neq 0$ with $\sum_{j=1}^n c_j(\partial\rho_\nu/\partial\bar{z}_j) = 0$, $\nu = 1, \ldots, m$. In other words, the tangent vector to $\mathbb{C}^n$ at x_0,

$$\sum_{j=1}^n c_j \frac{\partial}{\partial \bar{z}_j},$$

annihilates $\rho_1, \ldots, \rho_m$, and hence by Exercise 17.2 $\sum$ has a complex tangent at x_0, which is contrary to assumption.

Hence $\xi_1, \ldots, \xi_m$ span $\mathbb{C}^n$, and so we can find $\nu_1, \ldots, \nu_n$ with $\xi_{\nu_1}, \ldots, \xi_{\nu_n}$ linearly independent. By continuity, then, the vectors

$$\left(\frac{\partial\rho_{\nu_j}}{\partial\bar{z}_1}, \ldots, \qquad \frac{\partial\rho_{\nu_j}}{\partial\bar{z}_n}\right)_x, \qquad j = 1, \ldots, n$$

are linearly independent, and so form a basis for $\mathbb{C}^n$, for all x in some neighborhood of x_0. This was the assertion.

Relabel $\rho_{\nu_1}, \ldots, \rho_{\nu_n}$ to read $\rho_1, \ldots, \rho_n$. Define functions $h_1, \ldots, h_n$ in ω_0 by

$$\left(\frac{\partial u}{\partial\bar{z}_1}, \ldots, \frac{\partial u}{\partial\bar{z}_n}\right)(x) = \sum_{i=1}^n h_i(x)\left(\frac{\partial\rho_i}{\partial\bar{z}_1}, \ldots, \frac{\partial\rho_i}{\partial\bar{z}_n}\right)_x, \qquad x \in \omega_0.$$

Solve for $h_i(x)$. All the coefficients in this $n \times n$ system of equations are of class $e - 1$, so $h_i \in C^{e-1}(\omega_0)$. We have

$$\bar{\partial}u = \sum_{i=1}^n h_i\bar{\partial}\rho_i \text{ in } \omega_0.$$

Put $u_1 = u - \sum_{i=1}^n h_i\rho_i$. So $u_1 = u$ on $\sum$, and

$$\bar{\partial}u_1 = \bar{\partial}u - \sum_{i=1}^n h_i\bar{\partial}\rho_i - \sum_{i=1}^n \bar{\partial}h_i \cdot \rho_i = -\sum_{i=1}^n \bar{\partial}h_i \cdot \rho_i.$$

In the same way in which we got the h_i, we can find functions h_{ij} in $C^{e-2}(\omega_0)$ with

$$\bar{\partial}h_i = \sum_{j=1}^n h_{ij}\bar{\partial}\rho_j, i = 1, \ldots, n.$$

Since $\bar{\partial}\rho_1, \ldots, \bar{\partial}\rho_n$ are linearly independent at each point of ω_0, the same is true of the (0, 2)-forms $\bar{\partial}\rho_j \wedge \bar{\partial}\rho_i$ with $i < j$.

$$\begin{aligned} 0 = \bar{\partial}^2 u = \bar{\partial}\left(\sum_{i=1}^n h_i\bar{\partial}\rho_i\right) &= \sum_i \left(\sum_j h_{ij}\bar{\partial}\rho_j\right) \wedge \bar{\partial}\rho_i \\ &= \sum_{i<j}(h_{ij} - h_{ji}) \cdot \bar{\partial}\rho_j \wedge \bar{\partial}\rho_i \end{aligned}$$

Hence $h_{ij} = h_{ji}$ for $i < j$. Put

$$u_2 = u_1 + \frac{1}{2!} \sum_{i,j} h_{ij} \rho_i \rho_j.$$

So $u_2 = u$ on $\sum$ and

$$\bar{\partial} u_2 = -\sum_i \bar{\partial} h_i \cdot \rho_i + \frac{1}{2!} \sum_{i,j} \bar{\partial}(h_{ij} \rho_i \rho_j + R,$$

where

$$R = \frac{1}{2} \sum_{i,j} h_{ij} \rho_i \bar{\partial} \rho_j + \frac{1}{2} \sum_{i,j} h_{ij} \rho_j \bar{\partial} \rho_i$$

$$= \frac{1}{2} \sum_i \bar{\partial} h_i \cdot \rho_i + \frac{1}{2} \sum_j \bar{\partial} h_j \rho_j,$$

so

$$\bar{\partial} u_2 = \frac{1}{2!} \sum_{i,j} \bar{\partial} h_{ij} \cdot \rho_i \rho_j.$$

We define inductively functions h_I on ω_0, I a multiindex, by

$$\bar{\partial} h_i = \sum_{i=1}^{n} h_{Ij} \bar{\partial} \rho_j,$$

and we define functions u_N, $N = 1, 2, \ldots, e-1$, by

$$u_N = u_{N-1} + \frac{(-1)^N}{N!} \sum_{|I|=N} h_I \rho_I,$$

where $I = (\beta_1, \ldots, \beta_n)$, $|I| = \sum \beta_i$, $\rho_I = \rho_1^{\beta_1} \cdots \rho_n^{\beta_n}$. Then $h_I \in C^{e-N}(\omega_0)$ if $|I| = N$, and $u_N \in C^{e-N}(\omega_0)$.

We verify

$$\bar{\partial} u_N = \frac{(-1)^N}{N!} \sum_{|I|=N} \bar{\partial} h_I \cdot \rho_I, \text{ for each } N.$$

By slightly shrinking ω_0 we get a constant C such that $|\rho_I(z)| \leq C d(z)^N$ in ω_0 if $|I| = N$, and hence there is a constant C_1 with

$$\left| \frac{\partial u_N}{\partial \bar{z}_j}(z) \right| \leq C_1 d(z)^N, \qquad j = 1, \ldots, n,\ z \in \omega_0.$$

In particular, $u_{e-1} \in C^1(\omega_0)$, $u_{e-1} = u$ on $\sum$, and

$$\left| \frac{\partial u_{e-1}}{\partial \bar{z}_j} \right| \leq C_1 d(z)^{e-1}, \qquad C_1 \text{ depending on } \omega_0.$$

Also, $u = 0$ on an open subset of ω_0 implies that $u_{e-1} = 0$ there.

For each $x_0 \in K$ we now choose a neighborhood ω_{x_0} in $\mathbb{C}^n$ of the above type. Finitely many of these neighborhoods, say, $\omega_1, \ldots, \omega_g$, cover K.

Choose $\chi_1, \ldots, \chi_g \in C^\infty(\mathbb{C}^n)$ with supp $\chi_\alpha \subset \omega_\alpha, 0 \leq \chi_\alpha \leq 1$, and $\sum_{\alpha=1}^{g} \chi_\alpha = 1$ on K.

By the above construction, applied to $\chi_\alpha u$ in place of u, choose U_α in $C^1(\omega_\alpha)$ with $U_\alpha = \chi_\alpha u$ in $\sum \cap \omega_\alpha$, supp $U_\alpha \subset$ supp $\chi_\alpha u$, and

$$(*) \qquad \left| \frac{\partial U_\alpha}{\partial \bar{z}_j}(z) \right| \leq C_\alpha \cdot d(z)^{e-1}, \qquad z \in \omega_\alpha,\ j = 1, \ldots, n.$$

Since supp $U_\alpha \subset \omega_\alpha$, we can define $U_\alpha = 0$ outside ω_α to get a C^1-function in the whole space, and (*) holds for all z in $\mathbb{C}^n$.

Put $U = \sum_{\alpha=1}^{g} U_\alpha$. Then $U \in C^1(\mathbb{C}^n)$, and for $z \in K$,

$$U(z) = \sum_{\alpha=1}^{g} U_\alpha(z) = \sum_{\alpha=1}^{g} \chi_\alpha(z) u(z) = u(z) \sum_\alpha \chi_\alpha = u(z).$$

For every z,

$$\frac{\partial U}{\partial \bar{z}_j}(z) = \sum_{\alpha=1}^{g} \frac{\partial U_\alpha}{\partial \bar{z}_j}(z),$$

so, by (*),

$$\left| \frac{\partial U}{\partial \bar{z}_j}(z) \right| \leq g \cdot C \cdot d(z)^{e-1}, \qquad \text{where } C = \max_{1 \leq \alpha \leq g} C_\alpha.$$

This completes step 2. □

Proof of Theorem 17.1. It remains to carry out step 3.

Without loss of generality, $\sum$ is an open subset of some smooth k-dimensional manifold $\sum_1$ such that the closure of $\sum$ is a compact subset of $\sum_1$. It follows that the $2n$-dimensional volume of the ε-tube around $\sum$, i.e, $\{x \in \mathbb{C}^n | d(x), \varepsilon\}$, $= O(\varepsilon^{2n-k})$ as $\varepsilon \to 0$.

Fix ε and choose the set of ω_ε by Lemma 17.3. By (6), $\omega_\varepsilon \subset \varepsilon$-tube around $\sum$, so the volume of $\omega_\varepsilon = O(\varepsilon^{2n-k})$.

By (8), (9), and (10), Theorem 16.1 may be applied to ω_ε, where we take $\rho = u_\varepsilon$.

Given that u is in $C^\infty(\mathbb{C}^n)$, by Lemma 17.4 with $K = X$ we can find U_ε in $C^1(\mathbb{C}^n)$ such that for all z and j,

$$\left| \frac{\partial U_\varepsilon}{\partial \bar{z}_j} \right| \leq C d(z)^{e-1} \qquad \text{and} \qquad U_\varepsilon = u \text{on } X.$$

By (6) this implies

$$(15) \qquad \left| \frac{\partial U_\varepsilon}{\partial \bar{z}_j} \right| \leq C \varepsilon^{e-1} \text{in } \omega_\varepsilon.$$

Put $g = \bar\partial U_\varepsilon$. Then $\bar\partial g = 0$ in ω_ε. By Theorem 16.1, $\exists V_\varepsilon$ in $L^2(\omega_\varepsilon)$ such that, as distributions, $\bar\partial V_\varepsilon = g$; i.e.,

$$\frac{\partial V_\varepsilon}{\partial \bar z_j} = \frac{\partial U_\varepsilon}{\partial \bar z_j}, \quad \text{all } j, \tag{16}$$

and

$$\int_{\omega_\varepsilon} |V_\varepsilon|^2 dV \le C' \int_{\omega_\varepsilon} \left(\sum_{j=1}^n \left| \frac{\partial U_\varepsilon}{\partial \bar z_j} \right|^2 \right) dV. \tag{17}$$

Equations (15) and (17) and the volume estimate on ω_ε give

$$\int_{\omega_\varepsilon} |V_\varepsilon|^2 dV \le C'' \varepsilon^{2e-2+2n-k}. \tag{18}$$

By (16) and Lemma 16.8, V_ε is continuous in ω_ε. Further, fix $x \in X$ and put $B_x =$ ball of center x, radius $\varepsilon/2$. Lemma 16.8 implies that

$$|V_\varepsilon(x)| \le K \left\{ \varepsilon^{-n} ||V_\varepsilon|| L^2(B_x) + \varepsilon \sup_{B_x} \left(\max_j \left| \frac{\partial V\varepsilon}{\partial \bar z_j} \right| \right) \right\}. \tag{19}$$

But $B_x \subset \omega_\varepsilon$ by (7), so (18), (15), and (16) give

$$|V_\varepsilon(x)| \le K\{\varepsilon^{e-1-(k/2)} + \varepsilon^e\}, \tag{20}$$

where K is independent of x. Thus if $e > k/2 + 1$, $\sup_X |V_\varepsilon| \to 0$ as $\varepsilon \to 0$.

Step 3 is now complete. Theorem 17.1 is thus proved. □

As an application of Theorem 17.1, we consider the following problem: Let X be a compact subset of $\mathbb{C}^n$ and $f_1, \dots, f_k$ elements of $C(X)$. Let

$$[f_1, \dots, f_k | X]$$

denote the class of functions on X that are uniform limits on X of polynomials in $f_1, \dots, f_k$. The Stone-Weierstrass theorem gives

$$[z_1, \dots, z_n, \bar z_1, \dots, \bar z_n | X] = C(X).$$

We shall prove a perturbation of this fact. Let Ω be a neighborhood of X and let $R_1, \dots, R_n$ be complex-valued functions defined in Ω. Denote by R the vector-valued function $R = (R_1, \dots, R_n)$.

Theorem 17.5. *Assume that* $\exists k < 1$ *such that*

$$|R(z_1) - R(z_2)| \le k|z_1 - z_2| \qquad \text{if } z_1, z_2 \in \Omega \tag{21}$$

Assume also that each $R_j \in C^{n+2}(\Omega)$. *Then*

$$[z_1, \dots, z_n, \bar z_1 + R_1, \dots, \bar z_n + R_n | X] = C(X).$$

Note. Equation (21) is a condition on the Lipschitz norm of R. No such condition on the sup norm of R would suffice.

EXERCISE 17.4. Put X = closed unit disk in the z-plane and fix $\varepsilon > 0$. Show that $\exists$ a function Q, smooth in a neighborhood of X, with $|Q| \leq \varepsilon$ everywhere and $[z, \bar{z} + Q|X] \neq C(X)$.

Let Φ denote the map of Ω into $\mathbb{C}^{2n}$ defined by

$$\Phi(z) = (z, \bar{z} + R(z))$$

and let $\sum$ be the image of Ω under Φ. Evidently $\sum$ is a submanifold of an open set in $\mathbb{C}^{2n}$ of dimension $2n$ and class $n + 2$. Since $n + 2 > (2n/2) + 1$, the condition of "sufficient smoothness" holds for $\sum$.

Lemma 17.6. *$\sum$ has no complex tangents.*

PROOF. If $\sum$ has a complex tangent, then $\exists$ two tangent vectors to $\sum$ differing only by the factor i.

With $d\Phi$ denoting the differential of the map Φ, we can hence find $\xi, \eta \in \mathbb{C}^n$ different from 0 so that at some point of Ω,

$$d\Phi(\eta) = i d\Phi(\xi). \tag{22}$$

Let R_z denote the $n \times n$ matrix whose (j, k)th entry is $\partial R_j / \partial z_k$ and define $R_{\bar{z}}$ similarly. For any vector α in $\mathbb{C}^n$,

$$d\Phi(\alpha) = (\alpha, \bar{\alpha} + R_z\alpha + R_{\bar{z}}\bar{\alpha}).$$

Hence (22) gives

$$(\eta, \bar{\eta} + R_z\eta + R_{\bar{z}}\bar{\eta}) = i(\xi, \bar{\xi} + R_z\xi + R_{\bar{z}}\bar{\xi}).$$

It follows that $\eta = i\xi$ and

$$\bar{\xi} + R_{\bar{z}}\bar{\xi} = 0. \tag{23}$$

By Taylor's formula, for $z \in \Omega$, $\theta \in \mathbb{C}^n$, and ε real,

$$R(z + \varepsilon\theta) - R(z) = R_z\varepsilon\theta + R_{\bar{z}}\varepsilon\bar{\theta} + o(\varepsilon).$$

Applying (21) with $z_1 = z + \varepsilon\theta$, $z_2 = z$, and letting $\varepsilon \to 0$ then gives

$$|R_z\theta + R_{\bar{z}}\bar{\theta}| \leq k|\theta|. \tag{24}$$

Replacing θ by $i\theta$ gives

$$|R_z\theta - R_{\bar{z}}\bar{\theta}| \leq k|\theta|. \tag{24$'$}$$

Equations (24) and (24′) together give

$$|R_{\bar{z}}\bar{\theta}| \leq k|\theta| \qquad \text{for all } \theta \in \mathbb{C}^n, \tag{25}$$

and this contradicts (23). Thus $\sum$ has no complex tangent $\square$

Lemma 17.7. *$\Phi(X)$ is a polynomially convex compact set in $\mathbb{C}^{2n}$.*

PROOF. Put $\mathfrak{A} = [z_1, \ldots, z_n, \bar{z}_1 + R_1, \ldots, \bar{z}_n + R_n | X]$,

$$\mathfrak{A}_1 = [z_1, \ldots, z_{2n} | X_1], \qquad \text{where } X_1 = \Phi(X).$$

The map Φ induces an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}_1$. To show that X_1 is polynomially convex is equivalent to showing that every homomorphism of $\mathfrak{A}_1$ into $\mathbb{C}$ is evaluation at a point of X_1, and so to the corresponding statement about $\mathfrak{A}$ and X.

Let h be a homomorphism of $\mathfrak{A}$ into $\mathbb{C}$. Choose. by Exercise 1.2, a probability measure μ on X so that

$$h(f) = \int_X f du, \qquad \text{all } f \in \mathfrak{A}.$$

Put $h(z_i) = \alpha_i, i = 1, \ldots, n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Choose an extension of R to a map of $\mathbb{C}^n$ to $\mathbb{C}^n$ such that (21) holds whenever $z_1, z_2 \in \mathbb{C}^n$. This can be done by a result of F. A. Valentine, A Lipschitz condition preserving extension of a vector function, *Am. J. Math.* **67** (1945).

Define for all $z \in X$,

$$f(z) = \sum_{i=1}^{n} (z_i - \alpha_i)((\bar{z}_i + R_i(z)) - (\bar{\alpha}_i + R_i(\alpha))).$$

Since z_i and $\bar{z}_i + R_i(z) \in \mathfrak{A}$ and α_i and $R_i(\alpha)$ are constants, $f \in \mathfrak{A}$. Evidently $h(f) = 0$. Also, for $z \in X$,

$$f(z) = \sum_{i=1}^{n} |z_i - \alpha_i|^2 + \sum_{i=1}^{n} (z_i - \alpha_i)(R_i(z) - R_i(\alpha)).$$

The modulus of the second sum is $\leq |z - \alpha||R(z) - R(\alpha)| \leq k|z - \alpha|^2$, by (21). Hence $\operatorname{Re} f(z) \geq 0$ for all $z \in X$, and $\operatorname{Re} f(z) = 0$ implies that $z = \alpha$. Also,

$$0 = \operatorname{Re} h(f) = \int_X \operatorname{Re} f \, d\mu.$$

It follows that $\alpha \in X$ and that μ is concentrated at α. Hence h is evaluation at α, and we are done. □

PROOF OF THEOREM 17.5. We now know that $\Phi(X)$ is a polynomially convex compact subset of $\sum$ and that $\sum$ is a submanifold of $\mathbb{C}^{2n}$ without complex tangents. Theorem 17.1 now gives that $P(\Phi(X)) = C(\Phi(X))$, and this is the same as to say that

$$[z_1, \ldots, z_n \bar{z}_1 + R_1, \ldots, \bar{z}_n + R_n | X] = C(X).$$

□

NOTES

A result close to Theorem 17.1 was first announced by R. Nirenberg and R. O. Wells, Jr., Holomorphic approximation on real submanifolds of a complex manifold, *Bull. Am. Math. Soc.* **73** (1967), and a detailed proof was given the same authors in Approximation theorems on differentiable submanifolds of a complex manifold, *Trans. Am. Math. Soc.* **142** (1969). They follow a method of proof suggested by Hörmander. A generalization of Theorem 17.1 to certain cases where complex tangents may exist was given by Hörmander and Wermer in Uniform approximation on compact sets in $\mathbb{C}^n$, *Math. Scand.* **23** (1968). Theorem 17.5 is also proved in that paper, the case $n = 1$ of Theorem 17.5 having been proved earlier by Wermer in Approximation on a disk, *Math. Ann.* **155** (1964), under somewhat weaker hypotheses. Various other related problems are also discussed in the papers by Nirenberg and Wells and by Hörmander and Wermer. Further results in this area are due to M. Freeman. The proof of Lemma 17.4 is due to Nirenberg and Wells. (For recent work, see Wells [Wel].)

An elementary proof of Theorem 17.5, based on a certain integral transform, has recently been given by Weinstock in [Wei1].

18

Submanifolds of High Dimension

In Sections 13, 14 and 17 we have studied polynomial approximation on certain kinds of k-dimensional manifolds in $\mathbb{C}^n$. In this Section we consider the case $k > n$. Let $\sum$ be a k-dimensional submanifold of an open set in $\mathbb{C}^n$ with $n < k < 2n$. Let X be a compact set which lies on $\sum$ and contains a relatively open subset of $\sum$.

Lemma 18.1.

$$P(X) \neq C(X).$$

We first prove

Lemma 18.2. *Let S be a set in $\mathbb{C}^n$ homeomorphic to the n-sphere. Then $h(S) \neq S$.*

PROOF. $h(S) = \mathcal{M}(P(S))$. The algebra $P(S)$ has n generators and hence by Theorem 15.8 the n'th cohomology group of $\mathcal{M}(P(S))$ with complex coefficients vanishes. But $H^n(S, \mathbb{C}) \neq 0$. Hence $S \neq h(S)$. □

PROOF OF LEMMA 18.1. Choose a set $S \subset X$ with S homeomorphic to the n-sphere. By the last Lemma $h(S) \neq S$ and so $P(S) \neq C(S)$. Since an arbitrary continuous function on S extends to an element of $C(X)$, this implies $P(X) \neq C(X)$. □

We should like to explain the fact that arbitrary continuous functions on X cannot be approximated by polynomials, in terms of the geometry of $\sum$ as submanifold of $\mathbb{C}^n$.

Fix $x^0 \in \sum$ and a neighborhood U of x^0 on $\sum$. We shall try to construct an analytic disk E in $\mathbb{C}^n$ whose boundary lies in U. In other words, we seek a one-one continuous map Φ of $|z| \leq 1$ into $\mathbb{C}^n$ with Φ analytic in $|z| < 1$ and $\Phi(|z| = 1) \subset U$. We then take $E = \Phi(|z| \leq 1)$. Then every function approximable by polynomials uniformly on U extends analytically to E and hence $P(X) \neq C(X)$ whenever X contains U.

Example. Let $\sum$ be the 3-sphere $|z_1|^2 + |z_2|^2 = 1$ in $\mathbb{C}^2$ and fix $x^0 \in \sum$. Without loss of generality, $x^0 = (i, 0)$. We shall describe a family of analytic disks near x^0 each with its boundary lying on $\sum$.

Fix $t > 0$ and define the closed curve γ_t by:

$$z_1 = i\sqrt{1-t^2},\, z_2 = t\zeta,\, |\zeta| = 1.$$

γ_t lies on $\sum$ and bounds the analytic disk E_t defined:

$$z_1 = i\sqrt{1-t^2},\, z_2 = t\zeta,\, |\zeta| \leq 1.$$

As $t \to 0$, $\gamma_t \to x_0$.

We wish to generalize this example. Let $\sum^{2n-1}$ be a smooth (class 2) $(2n-1)$-dimensional hypersurface in some open set in $\mathbb{C}^n$ and fix $x^0 \in \sum^{2n-1}$. Let U be a neighborhood of x^0 on $\sum^{2n-1}$.

Theorem 18.3. *$\exists$ an analytic disk E whose boundary $\partial\exists$ lies in U.*

Note. After proving this theorem, we shall prove in Theorem 18.7 an analogous result for manifolds of dimension k with $n < k$. The method of proof will be essentially the same, and looking first at a hypersurface makes it easier to see the idea of the proof. By an affine change of complex coordinates we arrange that $x^0 = 0$ and that the tangent space to $\sum^{2n-1}$ at 0 is given by: $y_1 = 0$, where $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ are the real coordinates in $\mathbb{C}^n$. Then $\sum^{2n-1}$ is described parametrically near 0 by equations

$$(1).\qquad \begin{cases} z_1 &= x_1 + ih(x_1, w_2, \ldots, w_n) \\ z_2 &= w_2 \\ & \vdots \\ z_n &= w_n, \end{cases}$$

where $x_1 \in \mathbb{R}$, $(w_2, \ldots, w_n) \in \mathbb{C}^{n-1}$ and h is a smooth real valued function defined on $\times\mathbb{C}^{n-1}$ with h vanishing at 0 of order 2 or higher.

We need some definitions.

Definition 18.1. Put $\Gamma = \{\zeta || \zeta| = 1)\}$. A function f in $C(\Gamma)$ is a *boundary function* if $\exists F$ continuous in $|\zeta| \leq 1$ and analytic in $|\zeta| < 1$ with $F = f$ on Γ.

Given u defined on Γ, we put

$$\dot{u} = \frac{d}{d\theta}(u(e^{i\theta})).$$

Definition 18.2. H_1 is the space of all real-valued functions u on Γ such that u is absolutely continuous, $u \in L^2(\Gamma)$ and $\dot{u} \in L^2(\Gamma)$. For $u \in H_1$, we put

$$||u|_1 = ||u||_{L^2} + ||\dot{u}||_{L^2}.$$

Normed with $|| \ ||_1$, H_1 is a Banach space. Fix $u \in H_1$.

$$u = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

Since $\dot{u} \in L^2$, $\sum_1^\infty n^2(a_n^2 + b_n^2) < \infty$ and so $\sum_1^\infty (|a_n| + |b_n|) < \infty$.

Definition 18.3. For u as above,

$$Tu = \sum_{n=1}^{\infty} a_n \sin n\theta - b_n \cos n\theta.$$

Observe the following facts:

(2) If $u, v \in H_1$, then $u + iv$ is a boundary function provided $u = -Tv$.

(3) If $u \in H_1$, then $Tu \in H_1$ and $||Tu||_1 \le ||u||_1$.

Definition 18.4. Let $w_2, \dots, w_n$ be smooth boundary functions and put $w = (w_2, \dots, w_n)$. w is then a map of Γ into $\mathbb{C}^{n-1}$. For $x \in H_1$,

$$A_w x = -T\{h(x, w)\},$$

where h is as in (1). A_w is thus a map of H_1 into H_1.

Let U be as in Theorem 18.3 and choose $\delta > 0$ such that the point described by (1) with parameters x_1 and w lies in U provided $|x_1| < \delta$ and $|w_j| < \delta$, $2 \le j \le n$.

Lemma 18.4. *Let $w_2, \dots, w_n$ be smooth boundary functions with $|w_j| < \delta$ for all j and such that w_2 is schlicht, i.e., its analytic extension is one-one in $|\zeta| \le 1$. Put $A = A_w$. Suppose $x^* \in H_1$, $|x^*| < \delta$ on Γ and $Ax^* = x^*$. Then $\exists$ analytic disk E with ∂E contained in U.*

Proof. Since $Ax^* = x^*$, $x^* = -T\{h(X^*, w)\}$, and so $x^* + ih(x^*, w)$ is a boundary function by (2). Let ψ be the analytic extension of $x^* + ih(x^*, w)$ to $|\zeta| < 1$. The set defined for $|\zeta| \le 1$ by $z_1 = \psi(\zeta), z_2 = w_2(\zeta), \dots, z_n = w_n(\zeta)$ is an analytic disk E in $\mathbb{C}^n$. ∂E is defined for $|\zeta| = 1$ by $z_1 = x^*(\zeta) + ih(x^*(\zeta), w(\zeta))$, $z_2 = w_2(\zeta), \dots, z_n = w_n(\zeta)$ and so by (1) lies on $\sum^{2n-1}$. Since by hypothesis $|x^*| < \delta$ and $|w_j| < \delta$ for all j, $\partial E \subset U$. $\square$

In view of the preceding, to prove Theorem 18.3, it suffices to show that $A = A_w$ has a fix-point x^* in H_1 with $|x^*| < \delta$ for prescribed small w. To produce this fix-point, we shall use the following well-known Lemma on metric spaces.

Lemma 18.5. *Let K be a complete metric space with metric ρ and Φ a map of K into K which satisfies*

$$\rho(\Phi(x), \Phi(y)) \le \alpha\rho(x, y), \qquad \text{all } x, y \in K.$$

where α is a constant with $0 < \alpha < 1$. Then Φ has a fix-point in K.

We give the proof of Exercise 18.1.

As complete metric space we shall use the ball in H_1 of radius M, $B_M = \{x \in H_1 \big| ||x||_1 \leq M\}$. We shall show that for small M if $|w|$ is sufficiently small and $A = A_w$, then

(4) A maps B_M into B_M.

(5) $\exists\alpha, 0 < \alpha < 1,$ such that

$$||Ax - Ay||_1 \leq \alpha||x - y||_1 \qquad \text{for all } x, y \in B_M.$$

Hence Lemma 18.5 will apply to A.

We need some notation. Fix N and let $x = (x_1, \ldots, x_N)$ be a map of Γ into $\mathbb{R}^N$ such that $x_i \in H_1$ for each i.

$$\dot{x} = (\dot{x}_1, \ldots, \dot{x}_N), |x| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$||x||_1 = \sqrt{\int_\Gamma |x|^2 d\theta} + \sqrt{\int_\Gamma |\dot{x}|^2 d\theta}$$

$$||x||_\infty = \sup |x|, \text{ taken over } \Gamma.$$

Observe that $||x||_\infty \leq C||x||_1$, where C is a constant depending only on N. In the following two Exercises, h is a smooth function on $\mathbb{R}^N$ which vanishes at 0 of order ≥ 2.

*Exercise 18.2. $\exists$ constant K depending only on h such that for every map x of Γ into $\mathbb{R}^N$ with $||x||_\infty \leq 1$,

$$||h(x)||_1 \leq K(||x||_1)^2.$$

*Exercise 18.3. $\exists$ constant K depending only on h such that for every pair of maps x, y of Γ into $\mathbb{R}^N$ with $||x||_\infty \leq 1$, $||y||_\infty \leq 1$.

$$||h(x) - h(y)||_1 < K||x - y||_1(||x||_1 + ||y||_1).$$

Fix boundary functions $w_2, \ldots, w_n$ as earlier and put $w = (w_2, \ldots, w_n)$. Then w is a map of Γ into $\mathbb{C}^{n-1} = \mathbb{R}^{2n-2}$.

Lemma 18.6. *For all sufficiently small $M > 0$ the following holds: if $||w||_1 < M$ and $A = A_w$, then A maps B_M into B_M and $\exists\alpha$, $0 < \alpha < 1$, such that $||Ax - Ay||_1 \leq \alpha||x - y||_1$ for all $x, y \in B_M$.*

Proof. Fix M and choose w with $||w||_1 < M$ and choose $x \in B_M$. The map (x, w) takes Γ into $\mathbb{R} \times \mathbb{C}^{n-1} = \mathbb{R}^{2n-1}$. If M is small, $||(x, w)||_\infty \leq 1$. Since $(x, w) = (x, 0) + (0, w)$,

$$||(x, w)||_1 \leq ||(x, 0)||_1 + ||(0, w)||_1 = ||x||_1 + ||w||_1.$$

By Exercise 18.2,

$$\begin{aligned} ||h(x,w)||_1 &< K(||(x,w)||_1)^2 \\ &< K(||x||_1 + ||w||_1)^2 < K(M+M)^2 = 4M^2K. \\ ||Ax||_1 &= ||T\{h(x,w)\}||_1 \le ||h(x,w)||_1 < 4M^2K. \end{aligned}$$

Hence if $M < 1/4K$, $||Ax||_1 \le M$. So for $M < 1/4K$, A maps B_M into B_M.

Next fix $M < 1/4K$ and w with $||w||_1 < M$ and fix $x, y \in B_M$. If M is small, $||(x,w)||_\infty \le 1$ and $||(y,w)||_\infty \le 1$.

$$Ax - Ay = T\{h(y,w) - h(x,w)\}.$$

Hence by (3), and Exercise 18.3, $||Ax - Ay||_1 \le ||h(y,w) - h(x,w)||_1 \le K||(x,w) - (y,w)||_1(||x,w)||_1 + ||(y,w)||_1) \le K||x-y||_1(||x||_1 + ||y||_1 + 2||w||_1) \le 4MK||x-y||_1$. Put $\alpha = 4MK$. Then $\alpha < 1$ and we are done. □

Proof of Theorem 18.3. Choose M by Lemma 18.6, choose w with $||w||_1 < M$ and put $A = A_w$. In view of Lemmas 18.5 and 18.6, A has a fix-point x^* in B_M. Since for $x \in H_1$, $||x||_\infty \le C||x||_1$, where C is a constant, for given $\delta > 0$ $\exists M$ such that $x^* \in B_M$ implies $|x^*| < \delta$ on Γ. By Lemma 18.4 it follows that the desired analytic disk exists. So Theorem 18.3 is proved. □

We now consider the general case of a smooth k-dimensional submanifold $\sum^k$ of $\mathbb{C}^n$ with $k > n$. Assume $0 \in \sum^k$. Denote by P the tangent space to $\sum^k$ at 0, regarded as a real-linear subspace of $\mathbb{C}^n$. Let Q denote the largest complex-linear subspace of P.

Exercise 18.4. $\dim_{\mathbb{C}} Q = k - n$.

Note. It follows that, since $k > n$, $\sum^k$ has at least one complex tangent at 0.

It is quite possible that $\dim_{\mathbb{C}} Q = k - n$. This happens in particular when Q is a complex-analytic manifold, for then $\dim_{\mathbb{C}} Q = k/2$, and $k/2 > k - n$ since $2n > k$.

We impose condition

$$\dim_{\mathbb{C}} Q = k - n. \tag{6}$$

Exercise 18.5. Assume (6) holds. For each x in $\sum^k$ denote by Q_x the largest complex linear subspace of the tangent space to $\sum^k$ at x. Show that $\dim_{\mathbb{C}} Q_x = k - n$ for all x in some neighborhood of 0.

Theorem 18.7. *Assume (6). Let U be a neighborhood of 0 on $\sum^k$. Then $\exists$ an analytic disk E whose boundary ∂E lies in U.*

Note. When $k = 2n-1$, $k - n = n - 1$ and since $\dim_{\mathbb{C}} Q \le n - 1$, Exercise 18.4 gives that $\dim_{\mathbb{C}} Q = n - 1$. So (6) holds. Hence Theorem 18.7 contains Theorem 18.3.

Lemma 18.8. *Assume (6). Then after a complex-linear change of coordinates $\sum^k$ can be described parametrically near 0 by equations*

$$
(7) \qquad \begin{cases}
z_1 & = x_1 + ih_1(x_1, \dots, x_{2n-k}, w_1, \dots, w_{k-n}) \\
z_2 & = x_2 + ih_2(x_1, \dots, x_{2n-k}, w_1, \dots, w_{k-n} \\
 & \vdots \\
z_{2n-k} & = x_{2n-k} + ih_{2n-k}(x_1, \dots, x_{2n-k}, w_1, \dots, w_{k-n}) \\
z_{2n-k+1} & = w_1 \\
 & \vdots \\
z_n & = w_{k-n}
\end{cases}
$$

where $x_1, \dots, x_{2n-k} \in \mathbb{R}$, $w_q, \dots, w_{k-n} \in \mathbb{C}$ and $h_1, \dots, h_{2n-k}$ are smooth real-valued functions defined on $\mathbb{R}^{2n-k} \times \mathbb{C}^{k-n} = \mathbb{R}^k$ in a neighborhood of 0 and vanishing at 0 of order ≥ 2.

Proof. Put $z_j = x_j + iy_j$ for $1 \leq j \leq n$. The tangent space P to $\sum^k$ at 0 is defined by equations:

$$\sum_{j=1}^{n} a_j^\nu x_j + b_j^\nu y_j = 0, \qquad \nu = 1, 2, \dots, 2n - k$$

where a_j^ν, b_j^ν are real constants. We chose complex linear functions

$$L^\nu(z) = \sum_{j=1}^{n} c_j^\nu z_j, \qquad \nu = 1, 2, \dots, 2n - k$$

where c_j^ν are complex constants such that $\sum_{j=1}^n a_j^\nu x_j + b_j^\nu y_j = \operatorname{Im} L^\nu(z)$ for each ν. So P is given by the equations:

$$\operatorname{Im} L^\nu(z) = 0, \qquad \nu = 1, 2, \dots, 2n - k.$$

We claim that $L^1, \dots, L^{2n-k}$ are linearly independent functions over $\mathbb{C}$.

For consider the set $Q_1 = \{z \in \mathbb{C}^n | L^\nu(z) = 0 \text{ for all } \nu\}$. Q_1 is a complex linear subspace of P. If the L^ν were dependent, $\dim_{\mathbb{C}} Q_1 > n - (2n - k) = k - n$, contradicting (6). So they are independent. We define new coordinates $Z_1, \dots, Z_n$ in $\mathbb{C}^n$ by a linear change of coordinates such that $Z_\nu = L^\nu$ for $\nu = 1, \dots, 2n - k$. Put $Z_\nu = X_\nu + iY_\nu$. Then P has the equations

$$Y_1 = Y_2 = \cdots = Y_{2n-k} = 0.$$

Without loss of generality, then, P is given by equations:

$$(8) \qquad y_1 = y_2 = \cdots = y_{2n-k} = 0.$$

Let $x_j = x_j(t)$, $y_j = y_j(t)$, $1 \leq j \leq n$, $t \in \mathbb{R}^k$, be a local parametric representation of $\sum^k$ at 0 with $t = 0$ corresponding to 0. Since P is given by (8), at $t = 0$ $\partial y_j/\partial t_1 = \partial y_j/\partial t_2 = \cdots = \partial y_j/\partial t_k = 0$, $j = 1, 2, \dots, 2n - k$. Since the Jacobian of the map:

$$t \to (x_1(t), y_1(t), \dots, x_n(t), y_n(t))$$

at $t = 0$ has rank k, it follows that the determinant

$$\left| \begin{matrix} \frac{\partial x_1}{\partial t_1} & \cdots & \frac{\partial x_1}{\partial t_k} \\ \vdots & & \\ \frac{\partial x_n}{\partial t_1} & \cdots & \frac{\partial x_n}{\partial t_k} \\ \frac{\partial y_{2n-k+1}}{\partial t_1} & \cdots & \frac{\partial y_{2n-k+1}}{\partial t_k} \\ \vdots & & \\ \frac{\partial y_n}{\partial t_1} & \cdots & \frac{\partial y_n}{\partial t_k} \end{matrix} \right|_{t=0} \neq 0.$$

Hence we can solve the system of equations:

$$\begin{aligned} x_1 &= x_1(t) \\ &\vdots \\ x_n &= x_n(t) \\ y_{2n-k+1} &= y_{2n-k+1}(t), \qquad t = (t_1, \ldots, t_k) \\ &\vdots \\ y_n &= y_n(t) \end{aligned}$$

for $t_1, \ldots, t_k$ in terms of $x_1, \ldots, x_n, y_{2n-k+1}, \ldots, y_n$ locally near 0. Let us put

$$\begin{aligned} u_1 &= x_{2n-k+1}, \ldots, u_{k-n} = x_n, \\ v_1 &= y_{2n-k+1}, \ldots, v_{k-n} = y_n. \end{aligned}$$

Put $x = (x_1, \ldots, x_{2n-k}$, $u = (u_1, \ldots, u_{k-n})$, $v = (v_1, \ldots, v_{k-n})$. Then parametric equations for $\sum^k$ at 0 can be written:

$$\begin{aligned} x_1 &= x_1 \\ y_1 &= h_1(x, u, v) \\ &\vdots \\ x_{2n-k} &= x_{2n-k} \\ y_{2n-k} &= h_{2n-k}(x, u, v) \\ x_{2n-k+1} &= u_1 \\ y_{2n-k+1} &= v_1 \\ &\vdots \\ x_n &= u_{k-n} \\ y_n &= v_{k-n}, \end{aligned}$$

where each h_j is a smooth function on $\mathbb{R}^k$, in a neighborhood of 0. In view of (8), each h_j vanishes at 0 of order ≥ 2. Setting

$$u_j + iv_j = w_j, \qquad j = 1, 2, \ldots, k - n,$$

we obtain (7). $\square$

We sketch the proof of Theorem 18.7:

With $h_1, \ldots, h_{2n-k}$ as in (7), we put

$$h(x, w) = (h_1(x, w), \ldots, h_{2n-k}(x, w)).$$

h is a map defined on a neighborhood of 0 in $\mathbb{R}^k$ and taking values in $\mathbb{R}^{2n-k}$. We shall use this vector-valued function h in the same way as we used the scalar-valued function h of (1) in proving Theorem 18.3.

Fix smooth boundary functions $w_1, \ldots, w_{k-n}$ on Γ such that w_1 is schlicht and put $w = (w_1, \ldots, w_{k-n})$. We seek a map $x^* = (x_1^*, \ldots, x_{2n-k}^*)$ of $\Gamma \to \mathbb{R}^{2n-k}$ such that

$$x^* + ih(x^*, w)$$

admits an analytic extension $\psi = (\psi_1, \ldots, \psi_{2n-k})$ to $|\zeta| < 1$ which takes values in $\mathbb{C}^{2n-k}$. Then the subset of $\mathbb{C}^n$ defined for $|\zeta| \leq 1$ by

$$z_1 = \psi_1(\zeta), \ldots, z_{2n-k} = \psi_{2n-k}(\zeta), z_{2n-k=1} = w_1(\zeta), \ldots, z_n = w_{k-n}(\zeta)$$

is an analytic disk E in $\mathbb{C}^n$ whose boundary ∂E is defined for $|\zeta| = 1$ by

$$z_1 = x_1^* + ih_1(x^*, w), \ldots, z_{2n-k} = x_{2n-k}^* + ih_{2n-k}(x^*, w),$$
$$z_{2n-k+1} = w_1, \ldots, z_n = w_{k-n}$$

and so in view of (7), ∂E lies on $\sum^k$.

We construct the desired x^* by a direct generalization of the proof given for Theorem 18.3. In particular we extend the definition 18.3 of the operator T to vector-valued functions $u = (u_1, \ldots, u_s)$ by: $Tu = (Tu_1, \ldots, Tu_s)$. We omit the details.

What can be said if $\sum$ is a smooth k-dimensional manifold in $\mathbb{C}^n$ with $k = n$? It is clear that no full generalization of Theorem 18.7 is possible in this case, since the real subspace $\sum_R$ of $\mathbb{C}^n$ is such a submanifold and there does not exist any analytic disk in $\mathbb{C}^n$ whose boundary lies on $\sum_R$.

What if $\sum$ is a compact orientable n-dimensional submanifold of $\mathbb{C}^n$? When $n = 1$, this means that $\sum$ is a simple closed curve in $\mathbb{C}$ and so $\sum$ is itself the boundary of an analytic disk in $\mathbb{C}$. When $n > 1$, we still see by reasoning as in the proof of Lemma 18.2 that $h(\sum) \neq \sum$. However, there need not exist any point $p \in \sum$ with the property that every neighborhood of p on $\sum$ contains the boundary of some analytic disk. This happens, in particular, when $\sum$ is the torus: $|z| = 1, |w| = 1$ in $\mathbb{C}^2$. This torus contains infinitely many closed curves which bound analytic disks in $\mathbb{C}^2$, but these curves are all "large."

A striking result, due to Bishop, [9], is that if $\sum$ is a smooth 2-sphere in $\mathbb{C}^2$, i.e., a diffeomorphic image of the standard 2-sphere, satisfying a mild restriction, then $\sum$ contains at least two points p such that every neighborhood of p on $\sum$ contains the boundary of some analytic disk.

NOTES

This section is due to E. Bishop, Differentiable manifolds in complex Euclidean space, *Duke Math Jour.* **32** (1965).

Given a k-dimensional smooth compact manifold $\sum$ in $\mathbb{C}^n$, it can occur that there exists a fixed open set $\mathcal{O}$ in $\mathbb{C}^n$ such that every function analytic in a neighborhood of $\sum$, no matter how small, extends to an analytic function in $\mathcal{O}$. This phenomenon for $k = 2n - 1$ was discovered by Hartogs. For $k = 4$, $n = 3$ an example of this phenomenon was given by Lewy in [Lew] and treated in general by Bishop in his above mentioned paper, as an application of the existence of the analytic disks he constructs. Substantial further work on this problem has been done. We refer to the discussion in Section 4 of R. O. Wells' paper, Function theory on differentiable submanifolds, *Contributions to Analysis*, a collection of papers dedicated to Lipman Bers, Academic Press (1974), and to the bibliography at the end of Wells' paper.

In the present Section we studied the problem of the existence of analytic varieties of complex dimension one whose boundary lies on a given manifold $\sum$. What can be said about the existence of analytic varieties of dimension greater than one whose boundary lies on $\sum$? In particular, let M^{2k-1} be a smooth odd-dimensional orientable compact manifold in $\mathbb{C}^n$ of real dimension $2k - 1$. When is M^{2k-1} the boundary of a piece of analytic variety, i.e. when does there exist a manifold with boundary Y (possibly having a singular set) such that the boundary of Y is M^{2k-1} and $Y \setminus M^{2k-1}$ is a complex analytic variety of complex dimension k? When $k = 1$, M^{2k-1} is a closed Jordan curve and Y exists only in the case that M^{2k-1} fails to be polynomially convex and in that case Y is the polynomially convex hull of M^{2k-1}. This situation was in effect, treated in Chapter 12 above. For arbitrary integers k the problem was solved by R. Harvey and B. Lawson in [HarL2], [Har]. For $k > 1$ the relevant condition on M^{2k-1} is expressed in terms of the complex tangents to M^{2k-1}.

19

Boundaries of Analytic Varieties

Part 1

Let γ be a simple closed oriented curve in $\mathbb{C}^2$. Under what conditions does γ bound an analytic variety of complex dimension one? More precisely, when does there exist an analytic variety Σ in some open set in $\mathbb{C}^2$ such that the closure of Σ is compact and γ is the boundary of Σ? Here we take "boundary" in the sense of Stokes' Theorem; i.e.,

$$\int_{\gamma} \omega = \int_{\Sigma} d\omega,$$

for every smooth 2-form ω on $\mathbb{C}^2$. This is stronger than being a boundary in a point-set theoretical sense and in particular takes orientation into account.

We have studied a related question regarding the polynomial hull of γ, in Chapter 12. Here we shall use a method of Harvey and Lawson [HarL2] based on the Cauchy transform.

We assume that γ is $\mathcal{C}^2$-smooth. Let π denote the projection of $\mathbb{C}^2$ on $\mathbb{C}$ with

$$\pi(z, w) = z, \quad \forall z, w.$$

The image curve $\pi(\gamma) \subseteq \mathbb{C}$ is then a smooth curve in $\mathbb{C}$ with possible self-intersections. We assume, for simplicity, that the set Λ of self-intersections is finite, and that there exists a $\mathcal{C}^2$-function f on $\pi(\gamma) \setminus \Lambda$ such that γ admits the representation:

$$\eta = f(\zeta), \quad \zeta \in \pi(\gamma) \setminus \Lambda,$$

where (ζ, η) is a point in $\mathbb{C}^2$.

To obtain the necessary conditions, we first assume that a variety Σ as above exists. We seek to describe the points of Σ in terms of the data on the curve γ.

Consider a fixed component ω of $\mathbb{C} \setminus \pi(\gamma)$. Since Σ is an analytic one-dimensional set, $\pi^{-1}(\omega) = \{(z, w) \in \Sigma : z \in \omega\}$ lies over ω as an n- sheeted analytic cover for some integer $n \geq 0$. For $z \in \omega$, we denote by $w_1(z), \ldots, w_n(z)$ the points of Σ over z; the $w_j(z)$ are not in general single-valued analytic functions of z in ω. The set $\pi^{-1}(\omega)$ is then described by the equation

$$\prod_{j=1}^{n}(w - w_j(z)) = 0, \ z \in \omega. \tag{1}$$

If we expand the product on the left-hand side, we obtain the expression

$$w^n - A_1(z)w^{n-1} + A_2(z)w^{n-2} + \cdots + (-1)^n A_n(z),$$

where $A_1(z) = \sum_{j=1}^{n} w_j(z)$, $A_2(z) = \sum_{j<k} w_j(z)w_k(z)$, and so on. The coefficient functions $A_1, \ldots, A_n$ are thus the elementary symmetric expressions in $w_1, \ldots, w_n$. It follows that the A_j are single-valued analytic functions on ω.

Next we shall show that n, the number of "sheets" of Σ over ω, is the winding number $n(\pi(\gamma), z)$ of the closed plane curve $\pi(\gamma)$ about z, for any point $z \in \omega$. Indeed this winding number is given by the integral

$$\frac{1}{2\pi i}\int_{\pi(\gamma)} \frac{d\zeta}{\zeta - z},$$

which is equal to the integral

$$\frac{1}{2\pi i}\int_{\gamma} \frac{d\zeta}{\zeta - z}.$$

We can evaluate the integral by applying the residue theorem on Σ to the form $\frac{d\zeta}{\zeta - z}$. We may assume that Σ does not branch over z. We conclude that the integral is just $1 + \cdots + 1$ (n terms) $= n$.

We shall next calculate the product in (1) for (z, w) with $z \in \omega$, $w \in \mathbb{C}$, with $|w|$ large, from the data on the curve γ. We write (ζ, η) for the coordinates of an arbitrary point on γ. Choose $R > 0$ such that $R > |\eta|$ for each $(\zeta, \eta) \in \gamma$. For w such that $|w| > R$, $\log(1 - \eta/w)$ is then well-defined for each $(\zeta, \eta) \in \gamma$. We have

$$\log(w - \eta) = \log(w) + \log(1 - \eta/w) \tag{2}$$

for all $(\zeta, \eta) \in \gamma$, where $\log w$ is defined up to an integer multiple of $2\pi i$. We set

$$\Phi(z, w) = \frac{1}{2\pi i}\int_{\gamma} \frac{\log(w - \eta)}{\zeta - z}\, d\zeta, \tag{3}$$

for $z \in \mathbb{C} \setminus \pi(\gamma)$, $|w| > R$. Then

$$\Phi(z, w) = (\log w)\frac{1}{2\pi i}\int_{\gamma} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma} \frac{\log(1 - \eta/w)}{\zeta - z}\, d\zeta \tag{4}$$

$$= n\log(w) + \frac{1}{2\pi i}\int_{\gamma} \frac{\log(1 - \eta/w)}{\zeta - z}\, d\zeta,$$

where n enters here as the winding number of $\pi(\gamma)$ about the point z.

Fix w, $|w| > R$. We calculate the integral in (4) by applying again the residue theorem to Σ. We assume first that Σ does not branch over z. Then the n functions

w_j are locally analytic near z and we get

$$\Phi(z, w) = n \log(w) + \sum_{j=1}^{n} \log(1 - \frac{w_j(z)}{w}). \tag{5}$$

By continuity, (5) holds as well if Σ branches over z.

We now define

$$F(z, w) = e^{\Phi(z,w)}, \quad z \in \omega, |w| > R. \tag{6}$$

Even though $\log(w)$ in (4) is only defined up to an integer multiple of $2\pi i$, $F(z, w)$ is unambiguously defined. Combining (5) and (6), we get

$$\begin{aligned} F(z, w) &= w^n \prod_{j=1}^{n} (1 - \frac{w_j(z)}{w}) = \prod_{j=1}^{n} (w - w_j(z)) \\ &= w^n - A_1(z) w^{n-1} + A_2(z) w^{n-2} + \cdots + (-1)^n A_n(z) \end{aligned} \tag{7}$$

for $z \in \omega$, $|w| > R$.

Thus F is a single-valued analytic function in $\omega \times \{|w| > R\}$ that extends to be analytic on $\omega \times \mathbb{C}$ as a monic polynomial in w of degree $n = n(\pi(\gamma), z)$ with coefficients being bounded analytic functions in ω. Moreover, this extension vanishes precisely on $\Sigma \cap \pi^{-1}(\omega)$.

One further consequence of the existence of Σ is the following condition: Let Ω be a polydisk containing γ, and let $\psi(\zeta, \eta)$ and $\sigma(\zeta, \eta)$ be analytic functions on Ω. Then

$$\int_\gamma \psi(\zeta, \eta) d\zeta + \sigma(\zeta, \eta) d\eta = 0. \tag{8}$$

This is because Σ must be contained in Ω, and so $\psi(\zeta, \eta) d\zeta + \sigma(\zeta, \eta) d\eta$ is a holomorphic one-form on $\Sigma \cup \gamma$ with $\gamma = b\Sigma$. It is clear that it is equivalent to say that the integral vanishes if ψ and σ are replaced by polynomials ζ and η, i.e., that

$$\int_\gamma P(\zeta, \eta) d\zeta + Q(\zeta, \eta) d\eta = 0 \tag{9}$$

for all polynomials P and Q. We shall refer to either (8) or (9) as the *moment condition on* γ. We thus have established the necessity part of the following result.

Theorem 19.1. *Let γ be an oriented simple closed curve in $\mathbb{C}^2$ with a finite number of self-intersections. Then a necessary and sufficient condition that there exists a bounded analytic variety Σ in $\mathbb{C}^2$ with $b\Sigma = \pm\gamma$ is that γ satisfies (9) (moment condition).*

The complete proof of this theorem involves a considerable number of technical details, and we shall refer the reader to the paper of Harvey and Lawson [HarL2] for these. Here we shall present a sketch that we hope conveys the essential aspects of the construction.

The orientation of an analytic variety in $\mathbb{C}^2$ is always taken to be the "natural" one induced by the complex structure. It clear that if a simple closed oriented curve γ satisfies the moment condition, then, if we reverse the orientation of γ, the moment condition is still satisfied. This explains the need for the "$\pm$" in the statement of the theorem.

We define U_0 to be the unbounded component of $\mathbb{C} \setminus \pi(\gamma)$ and denote by $U_1, U_2, \ldots$ the bounded components of $\mathbb{C} \setminus \pi(\gamma)$. We put

$$U = \bigcup_{j \geq 0} U_j,$$

that is, $U = \mathbb{C} \setminus \pi(\gamma)$. For each U_j, $j = 0, 1, \ldots$, set n_j equal to the winding number of $\pi(\gamma)$ about points of U_j. We have seen above that n_j also equals the number of sheets of Σ over U_j, a nonnegative integer. Thus we have $n_j \geq 0$, for $j = 0, 1, \ldots$.

Now we shall assume (8) and our objective is to produce an analytic variety Σ such that $\gamma = b\Sigma$, in the sense of Stokes' Theorem, after a possible change of orientation of γ.

Fix $R > 0$ as above. We define

$$\Phi(z, w) = \frac{1}{2\pi i} \int_\gamma \frac{\log(w - \eta)}{\zeta - z} d\zeta$$

for $z \in U$ and $|w| > R$, and also

$$F(z, w) = e^{\Phi(z,w)},$$

for $z \in U$ and $|w| > R$. For each i, $i = 0, 1, \ldots$, we define F_i as the restriction of F to $U_i \times \{|w| > R\}$. Thus each F_i is a single-valued non vanishing analytic function on $U_i \times \{|w| > R\}$. Splitting Φ in (4), we note that the second integral in (4) is analytic in w near ∞ and takes on the value 0 at $w = \infty$. Hence the Laurent decomposition of F_i has the form

$$F_i(z, w) = \sum_{k=-\infty}^{n_i} f_{ik}(z) w^k$$

for $(z, w) \in U_i \times \{|w| > R\}$, with f_{ik} holomorphic functions on U_i.

We need the following result: Let Ω^+ and Ω^- be two plane domains with common boundary arc α, where α is oriented positively with respect to Ω^+. (When Ω^+ and Ω^- are components of $\mathbb{C} \setminus \pi(\gamma)$, this means that as z moves from Ω^- to Ω^+ across α, the winding number of $\pi(\gamma)$ about z increases by 1.) Let g be a C^2-smooth function defined on α. Put

$$G^+(z) = \frac{1}{2\pi i} \int_\alpha \frac{g(\zeta) d\zeta}{\zeta - z}, \qquad z \in \Omega^+$$

and

$$G^-(z) = \frac{1}{2\pi i} \int_\alpha \frac{g(\zeta) d\zeta}{\zeta - z}, \qquad z \in \Omega^-.$$

See the book of Muskhelishvili [Mu] for a discussion of the following.

Plemelj's Theorem. *G^+ and G^- have continuous extensions to α, again denoted G^+ and G^-. On α we have*

$$G^+(z) - G^-(z) = g(z), \qquad z \in \alpha. \tag{10}$$

Lemma 19.2. *Let i, j be indices such that the regions U_i, U_j have a common smooth (open) boundary arc α, with α positively oriented for U_j. Then:*

(i) *F_i has a continuous extension to $(U_i \cup \alpha) \times \{|w| > R\}$ and F_j has a continuous extension to $(U_j \cup \alpha) \times \{|w| > R\}$;*

(ii) *$F_j(a, w) = (w - f(a))F_i(a, w)$, $a \in \alpha$, $|w| > R$, where $(a, f(a))$ is the unique point on γ over a.*

Proof. We note that the hypothesis, that α is positively oriented for U_j, is equivalent to the identity $n_j = n_i + 1$ for the winding numbers. We represent γ by the equation:

$$\eta = f(\zeta), \qquad \zeta \in \pi(\gamma).$$

Fix w with $|w| > R$. For $z \in U_i$ we have

$$\Phi(z, w) = \frac{1}{2\pi i}\int_\gamma \frac{\log(w - \eta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\pi(\gamma)} \frac{\log(w - f(\zeta))}{\zeta - z}\,d\zeta.$$

$\pi(\gamma)$ is the union of α and a complementary curve β. So

$$\Phi(z, w) = \frac{1}{2\pi i}\int_\alpha \frac{\log(w - f(\zeta))}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_\beta \frac{\log(w - f(\zeta))}{\zeta - z}\,d\zeta. \tag{11}$$

Now f is smooth on α and a is at a positive distance from β. It follows that the integral over β is continuous (across α) at a. Put

$$L_i = \lim_{z \in U_i \to a} \Phi(z, w), \ L_j = \lim_{z \in U_j \to a} \Phi(z, w).$$

Plemelj's theorem, combined with (11), gives (i) and

$$L_j - L_i = \log(w - f(a)).$$

Exponentiating, we get

$$\frac{\exp(L_j)}{\exp(L_i)} = w - f(a),$$

so $\exp L_j = (\exp L_i)(w - f(a))$. Thus

$$\lim_{z \in U_j \to a} F_j(z, w) = \lim_{z \in U_i \to a} F_i(z, w)(w - f(a)).$$

This gives (ii) and we are done. □

We continue with the notation of Lemma 19.2. Let Ω be a domain in the z-plane and α a boundary arc of Ω. Denote by $\mathfrak{A}$ the ring of functions analytic on Ω and continuous on $\Omega \cup \alpha$.

Lemma 19.3. *Let G be a function continuous on $(\Omega \cup \alpha) \times \{|w| > R\}$ and analytic on $\Omega \times \{|w| > R\}$, and let N be a nonnegative integer such that*

$$G(z, w) = \sum_{k=-\infty}^{N} g_k(z) w^k \qquad z \in \Omega \cup \alpha,\ |w| > R, \tag{12}$$

where each g_k lies in $\mathfrak{A}$. Assume that for each $a \in \alpha$, the function $w \mapsto G(a, w)$ is rational of order at most M, for some positive integer M. Then there exist functions

$$P(z, w) = \sum_{j=0}^{k} p_j(z) w^j, \qquad Q(z, w) = \sum_{j=0}^{l} q_j(z) w^j$$

with each $p_j \in \mathfrak{A}$, $q_j \in \mathfrak{A}$, such that $G = P/Q$ on $\Omega \times \{|w| > R\}$.

Proof. By shrinking α we may assume that there is an integer l, $0 \le l \le M$, such that, for each $a \in \alpha$, $G(a, w)$ can be written as a quotient of two relatively prime polynomials in w such that the denominator is always of degree exactly equal to l.

$\mathfrak{A}$ is an integral domain, and we form the field of quotients of $\mathfrak{A}$, denoted $\mathcal{F}$. The space $\mathcal{F}^{l+1}$ of $(l+1)$-tuples $(t_1, \ldots, t_{l+1})$ of elements of $\mathcal{F}$ is then an $(l+1)$-dimensional vector space over $\mathcal{F}$.

We denote by $\mathcal{W}$ the subspace of $\mathcal{F}^{l+1}$ spanned by the set of vectors

$$(g_{-i}, g_{-i-1}, \ldots, g_{-i-l}), \qquad i = 1, 2, \ldots,$$

where the g_n are the Laurent coefficients of G.

Claim. $\mathcal{W}$ has dimension $< l + 1$.

Proof of Claim. If $\dim \mathcal{W} \ge l + 1$, then we can choose $l + 1$ positive integers $i_1, i_2, \cdots, i_{l+1}$ such that the vectors

$$(g_{-i_\nu}, g_{-i_\nu-1}, \cdots, g_{-i_\nu-l}), \qquad \nu = 1, 2, \cdots, l + 1,$$

are linearly independent in $\mathcal{F}^{l+1}$. Then the determinant

$$D = \begin{vmatrix} g_{-i_1} & g_{-i_1-1} & \cdots & g_{-i_1-l} \\ g_{-i_2} & g_{-i_2-1} & \cdots & g_{-i_2-l} \\ \vdots & \vdots & \vdots & \vdots \\ g_{-i_{l+1}} & g_{-i_{l+1}-1} & \cdots & g_{-i_{l+1}-l} \end{vmatrix}$$

in $\mathfrak{A}$ is nonzero.

On the other hand, fix $a \in \alpha$. By the choice of l above, there exist $p_1^0, \ldots, p_k^0$ and $q_1^0, \ldots, q_l^0$ in $\mathbb{C}$ such that

$$\sum_{j=0}^{k} p_j^0 w^j = \Big(\sum_{s=-\infty}^{N} g_s(a) w^s\Big)\Big(\sum_{j=0}^{l} q_j^0 w^j\Big), \qquad |w| > R,$$

with $q_l^0 \ne 0$. Since the coefficients of w^j on the left-hand side vanish for $j < 0$, in particular for $j = -i_\nu$, $j = -i_\nu - 1, \cdots, j = -i_\nu - l$, $\nu = 1, 2, \ldots, l + 1$, we have the system of $l+1$ equations:

$$g_{-i_\nu}(a)q_0^0 + g_{-i_\nu-1}(a)q_1^0 + \cdots + g_{-i_\nu-l}(a)q_l^0 = 0, \qquad \nu = 1, 2, \ldots, l+1.$$

The coefficient matrix of this system has a vanishing determinant, since $q_l^0 \neq 0$. Thus $D(a) = 0$.

This holds for each $a \in \alpha$. Since D is analytic on Ω and continuous on $\Omega \cup \alpha$, it follows that $D = 0$ in $\mathfrak{A}$. This is a contradiction and so $\dim \mathcal{W} < l+1$, as claimed.

Because of the claim, there exists $(C_0, C_1, \cdots, C_l) \in \mathcal{F}^{l+1}$ with not all $C_j = 0$, such that

$$\text{(13)} \qquad g_{-i}C_0 + g_{-i-1}C_1 + \cdots + g_{-i-l}C_l = 0, \qquad i = 1, 2 \ldots.$$

Since each $C_j \in \mathcal{F}$, it follows by clearing the denominators that there exist $q_j \in \mathfrak{A}$, $j = 0, 1, \cdots, l$, not all zero, so that

$$\text{(14)} \qquad g_{-i}q_0 + g_{-i-1}q_1 + \cdots + g_{-i-l}q_l = 0, \qquad i = 1, 2 \ldots.$$

But (14) is equivalent to

$$\text{(15)} \qquad \Big(\sum_{-\infty}^{N} g_n w^n\Big)\Big(\sum_{j=0}^{l} q_j w^j\Big) = \sum_{j=0}^{k} p_j w^j,$$

for some functions $p_0, p_1, \cdots, p_k \in \mathfrak{A}$. This yields Lemma 19.3. □

Lemma 19.4. *$F(z, w) = 1$ for $z \in U_0$, $|w| > R$, where U_0, as earlier, is the unbounded component of $\mathbb{C} \setminus \pi(\gamma)$.*

Proof. Recall that R can be chosen so that γ is contained in the polydisk $\Omega = \{(\zeta, \eta) : |\zeta| < R, |\eta| < R\}$. Fix z, w in $\mathbb{C}$ such that $|z| > R$ and $|w| > R$. Then, with an appropriate choice of a branch of the logarithm,

$$\frac{\log(w - \eta)}{\zeta - z}$$

is an analytic function of (ζ, η) on Ω. By the moment condition (8), then,

$$\int_\gamma \frac{\log(w - \eta)}{\zeta - z}\, d\zeta = 0.$$

This holds for all z with $|z| > R$, and hence, by analytic continuation, for all $z \in U_0$. Thus $\Phi(z, w) = 0$, for $z \in U_0$, $|w| > R$. Hence $F(z, w) = 1$, for $z \in U_0$, $|w| > R$, as desired. □

Lemma 19.5. *Fix $i \geq 0$. Then F_i is the quotient of two polynomials in w with coefficients analytic for $z \in U_i$, $|w| > R$. In particular, F_i has a meromorphic continuation to $U_i \times \mathbb{C}$.*

Proof. The statement for $i = 0$ follows from Lemma 19.4.

Now consider the situation when U_k and U_j are adjacent components of $\mathbb{C} \setminus \pi(\gamma)$ with common boundary arc α. Suppose, for definiteness, that α is positively oriented with respect to U_j, that is, $n_j = n_k + 1$. We want to show that if the conclusion of the lemma holds for one of F_j, F_k, then it holds for the other. By Lemma 19.2 we know that both F_k and F_j have continuous extensions to α. Suppose that we know that F_j is rational in w on U_j. It follows by continuity that F_j is rational in w on α. By Lemma 19.2, $F_j(a, w) = (w - f(a))F_k(a, w)$, for all $a \in \alpha$, $|w| > R$. Therefore, $F_k(a, w) = F_j(a, w)/(w - f(a))$ is rational in w. Now Lemma 19.3 yields that F_k is rational in w on U_k. In the same way, one shows that if F_k is rational in w on U_k, then F_j is rational in w on U_j.

For any s we choose a sequence of indices

$$i_0 = 0, i_1, \ldots, i_{m-1}, i_m = s$$

such that $U_{i_{j-1}}$ and U_{i_j} share a common boundary arc for $j = 1, 2, \cdots, s$. Now starting from U_0 and applying the previous paragraph, it follows by induction on the length m of the sequence that F_s is rational in w on U_s. □

Definition 19.1. Let Ω be an open set in $\mathbb{C}^n$. A *holomorphic chain* of complex dimension k in Ω is a formal sum $\sum n_j V_j$, where the *branches* $\{V_j\}$ constitute a locally finite family of irreducible analytic subvarieties of complex dimension k in Ω and the n_j are nonzero integers, possibly negative.

A holomorphic chain can be thought of as an analytic variety with additional structure; namely, a holomorphic chain has branches V_j that carry a multiplicity $|n_j|$ and an orientation given by the sign of n_j. The variety Σ that we seek in Theorem 19.1 should be more precisely viewed as a holomorphic chain. The set of holomorphic chains in Ω forms an abelian group under addition of the formal sums giving the chains. If F is a meromorphic function on Ω, then we can associate a holomorphic chain of complex dimension $n - 1$ in Ω to F (also known as "the divisor of F"); this chain is the sum of branches of the zero set of F taken with positive orientation and appropriate multiplicity and the branches of the pole set of F taken with negative orientation and appropriate multiplicity.

SKETCH OF A PROOF OF THEOREM 19.1.

For each $j \geq 0$ we have

$$F_j(z, w) = \frac{P_j(z, w)}{Q_j(z, w)}$$

for $z \in U_j$, where P_j is a monic polynomial in w of degree N_j, say, and Q_j is a monic polynomial in w of degree Z_j, say. The "crossing over the edge" argument of Lemma 19.5 shows that $N_j - Z_j = n_j$, since both the difference $N_j - Z_j$ and the winding number n_j change by 1 when we cross an edge α. We let V_j be the holomorphic chain of complex dimension 1 associated to F_j in $U_j \times \mathbb{C}$. (This means that the zero set of P_j is taken with the positive orientation and with the multiplicity induced by the order of the zero, and that the zero set of Q_j is taken with the negative orientation and the appropriate multiplicity.)

Let α be an edge between U_j and U_k. We want to show that V_j and V_k "patch together" nicely over α. We can assume that α is positively oriented with respect to U_j, i.e., $n_j = n_k + 1$. We know that $P_j(z, w)$, $Q_j(z, w)$, $P_k(z, w)$, $Q_k(z, w)$ extend continuously in z to α. We first define an exceptional set of points E of α as the set of points $z \in \alpha$, where the discriminant of any of the four functions $P_j(z, w)$, $Q_j(z, w)$, $P_k(z, w)$, $Q_k(z, w)$ (as polynomials in w) vanishes, or where $P_j(z, w)$, $Q_j(z, w)$ are not relatively prime, or where $P_k(z, w)$, $Q_k(z, w)$ are not relatively prime .

Fix $z \in \alpha \setminus E$. Since $F_j(z, w) = (w - f(z))F_k(z, w)$, we get

$$\frac{P_j}{Q_j} = (w - f(z))\frac{P_k}{Q_k}$$

as rational functions of w. Therefore,

$$P_j(z, w)Q_k(z, w) = (w - f(z))P_k(z, w)Q_j(z, w),$$

and so the linear factor $(w - f(z))$ divides $P_j(z, w)Q_k(z, w)$. Hence there are two cases: (a) $Q_j(z, w) = Q_k(z, w)$ and $P_j(z, w) = (w - f(z))P_k(z, w)$ or (b) $Q_j(z, w)(w - f(z)) = Q_k(z, w)$ and $P_j(z, w) = P_k(z, w)$. These two cases are similar, and we shall treat case (a) in detail.

First we note that the fact that case (a) holds at z implies that it holds for $z' \in \alpha$ near z. Indeed, the linear factor $(w - f(z))$ divides $P_j(z, w)$. It cannot also divide $Q_k(z, w)$, for then it would divide $P_k(z, w)Q_j(z, w)$ and so it would also divide $P_k(z, w)$ or $Q_j(z, w)$. Hence $P_j(z, w)$, $Q_j(z, w)$ or $P_k(z, w)$, $Q_k(z, w)$ would not be relatively prime, contradicting the choice of the set E. Thus $(w - f(z))$ does not divide $Q_k(z, w)$. Hence $Q_k(z, f(z)) \neq 0$. Therefore, $Q_k(z', f(z')) \neq 0$ for $z' \in \alpha$ near z. Hence $(w - f(z'))$ divides $P_j(z', w)$ for $z' \in \alpha$ near z.

We have, since $Q_j = Q_k$ on α, that $Z_j = Z_k$. The corresponding coefficients of w in Q_k and Q_j are continuous near α and analytic off of α. It follows that the coefficients are analytic in a neighborhood of $z \in \alpha \setminus E$ in $\mathbb{C}$. It is then clear that the zero sets of Q_k and Q_j patch together to form a variety over α.

Now, for $z \in \bar{U}_j$ near a fixed point $a \in \alpha \setminus E$, we can factor $P_j(z, w)$ into N_j distinct linear factors in w with coefficients continuous in $\bar{U}_j$ and analytic in U_j. For $z \in \alpha$, one of these factors is $w - f(z)$. Hence one of the linear factors $L(z, w)$ (for $z \in U_j$) of $P_j(z, w)$ has $w - f(a)$ as its "boundary value" at $z \in \alpha$. Then we can form locally on $\bar{U}_j$ near a the function $\tilde{P}(z, w) = P_j(z, w)/L(z, w)$ and argue (as before for Q_k and Q_j) that the zero sets of $\tilde{P}$ and P_k "patch" over α to form an analytic variety. Summarizing, we get that, over a point $a \in \alpha \setminus E$, the union of the closures of V_j and V_k patch together to give varieties without boundary over α (N_k of them with positive orientation, Z_k with negative orientation) together with one variety (with positive orientation) with boundary $\{w = f(z)\}$ over α . This completes case (a). Case (b) is quite similar except that the one variety with boundary in $\{w = f(z)\}$ occurs with negative orientation.

Thus we have produced a global holomorphic chain Σ by patching the $\{V_j\}$. Aside from the exceptional points on the arcs α separating the components $\{U_j\}$, our construction shows that locally, the boundary of Σ is γ, in the sense of Stokes'

Theorem. For a complete proof, one still needs to show that the boundary of Σ in the sense of Stokes' Theorem is precisely equal to γ. In particular, one must discuss the exceptional points. For these rather technical issues, we refer to the original paper [HarL2].

Until now we have not used the hypothesis that γ is a *single* curve and we know only that Σ is a finite union of branches each with positive or negative orientation. Suppose, by way of contradiction, that Σ had more than one branch (irreducible component). Consider an arc α on the boundary of the unbounded component U_0. Then, since the winding number changes by ± 1 as we cross α, there is only one of the irreducible components of Σ that contains limit points on the part of γ over α. This means that one of the branches V of Σ has as its boundary a proper compact subset τ of γ. In particular, τ is contained in a Jordan arc. The maximum principle shows that V is contained in the polynomial hull of τ. But the proof of Theorem 12.4 (basically, the argument principle) implies that τ is polynomially convex. This is the desired contradiction!

Thus Σ is irreducible, i.e., it consists of a single branch whose boundary is contained in γ. From this one can deduce that, by reversing the orientation of Σ if necessary in order that the orientation of Σ be positive, $b\Sigma = \pm\gamma$. $\square$

Remark. As noted in the proof, the argument used in Theorem 19.1 applies when γ is only assumed to be a finite union of disjoint simple closed oriented curves, of course satisfying the moment condition. The conclusion is then that there exists a holomorphic chain V such that $bV = \gamma$.

Part 2

Theorem 19.1 is only a special case of a general result of Harvey and Lawson [HarL2] that characterizes the compact odd-dimensional oriented real manifolds M in $\mathbb{C}^n$ that bound analytic varieties V. These varieties are bounded sets such that $bV = M$ in the sense of Stokes' Theorem.

There is an obvious necessary condition that $bV = M$: Suppose that V has complex dimension $p > 1$; therefore, M has real dimension $2p - 1 > 1$. For each $z \in M$, the tangent space to M, $T_z(M)$, is a real linear space of dimension $2p - 1$.

Exercise 19.1. Show that $T_z(M)$ contains a complex subspace of complex dimension $p - 1$ for each $z \in M$. [Since the pair (V, M) is a manifold with boundary, V has a tangent space at $z \in M$ that, being the limit of the (complex!) tangent spaces of points of $V \setminus M$, is also a complex linear space. $T_z(M)$ is a subspace of real codimension 1 in this complex linear space.]

The complex subspace of $T_z(M)$ has the largest complex dimension possible for a subspace of a real linear space of real dimension $2p - 1$. This explains the following terminology.

Definition 19.2. M (of real dimension $2p-1$) is *maximally complex* if, for all $z \in M$, $T_z(M)$ contains a complex subspace of complex dimension $p-1$.

We saw above that M is maximally complex if it bounds some V as above. Another necessary condition (this one global) can be obtained from Stokes' Theorem. Let ω be a smooth $(p, p-1)$-form on $\mathbb{C}^n$ such that $\bar\partial\omega = 0$. Hence $d\omega = \bar\partial\omega + \partial\omega = \partial\omega$. Then, assuming that $bV = M$, Stokes' Theorem gives $\int_M \omega = \int_V d\omega = \int_V \partial\omega = 0$, since $\partial\omega$ is of type $(p+1, p-1)$ and so is identically zero on the complex p-dimensional manifold V.

Definition 19.3. M satisfies the *moment condition* if $\int_M \omega = 0$ for all $(p, p-1)$ forms ω such that $\bar\partial\omega = 0$.

EXERCISE 19.2. Let ω be a smooth $(1,0)$-form on $\mathbb{C}^2$. Then $\bar\partial\omega = 0$ if and only if $\omega = A dz_1 + B dz_2$, with A, B entire functions on $\mathbb{C}^2$.

Hence, for $n = 2$, Definition 19.3 coincides with (8).

For $p = 1$, maximal complexity on M is vacuous and the appropriate condition for the existence of V such that $bV = M$ is the moment condition on M, as we have indicated in Theorem 19.1. However, when $p > 1$, Harvey and Lawson [HarL2] showed that these two obviously necessary conditions on M are indeed equivalent, and each implies that there exists a V such that $bV = M$.

A complete proof of the Harvey–Lawson [HarL2] result involves a considerable number of technical details. In particular, as in the case when M is a real curve, the variety V must be taken as having an orientation and a multiplicity. Thus V should be viewed as a holomorphic chain. Moreover, there are technical problems in establishing the validity of Stokes' Theorem in the presence of unavoidable possible singularities of V and also of singularities in the way in which M bounds V (even when both M and V are smooth). Harvey and Lawson [HarL2] overcome these difficulties by working with currents, that is, they define a holomorphic chain to be a current and they take the statement $bV = M$ in the sense of currents. We shall not define currents here. Instead, we shall only give an incomplete discussion of a simple case of a three-dimensional manifold in $\mathbb{C}^3$.

Theorem 19.6. *Let M be a compact connected oriented three-dimensional submanifold of $\mathbb{C}^3$. Suppose that M is maximally complex or, equivalently, that M satisfies the moment condition. Then there exists a (bounded) holomorphic chain V of complex dimension 2 in $\mathbb{C}^3 \setminus M$ such that $bV = M$.*

Rather than give a complete proof of this theorem, we shall indicate how the details of the proof of Theorem 19.1 formally carry over to this higher dimensional case. A principal point here is that the Cauchy integral of the previous proof is replaced by the Bochner–Martinelli integral in this case.

We denote a point $Z \in \mathbb{C}^3$ by $Z = (z, w)$, where $z = (z_1, z_2) \in \mathbb{C}^2$, $w \in \mathbb{C}$, and $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ is the projection $\pi(Z) = z$. Let $\mathcal{M} = \pi(M) \subseteq \mathbb{C}^2$. Then

$\mathcal{M}$ is an immersed 3-manifold with an orientation inherited from M. We write $U = \mathbb{C}^2 \setminus \mathcal{M} = \cup_{j \geq 0} U_j$, where the U_j are the connected components of U and U_0 is the unbounded component.

Recall that the Bochner–Martinelli kernel was defined in (13.9). We shall make a minor change in notation and write $K_{BM}(\zeta, z)$ for the $(n, n-1)$-form (in ζ) defined in (13.9) multiplied by the appropriate normalizing constant (depending only on n), so that

$$\int_{S(z,r)} K_{BM}(\zeta, z) = 1,$$

where $S(z, r)$ denotes the sphere centered at z of radius $r > 0$. For the remainder of this chapter we shall take $n = 2$, so that $K_{BM}(\zeta, z)$ will be a (2, 1)-form in $\mathbb{C}^2$. The pull back $\pi^*(K_{BM}(\zeta, z))$ is a (2, 1)-form in $\mathbb{C}^3$. We can integrate these 3-forms over $\mathcal{M}$ and M, respectively

First we want to define the index $I(z, \mathcal{M})$ of a point $z \in \mathbb{C}^2 \setminus \mathcal{M}$ with respect to $\mathcal{M}$. This is the direct analogue of the winding number in the complex plane. We set

$$I(z, \mathcal{M}) = \int_{\mathcal{M}} K_{BM}(\zeta, z),$$

for $z \in \mathbb{C}^2 \setminus \mathcal{M}$. We claim that this is an integer. To see this, choose a small ball of radius r disjoint from $\mathcal{M}$ and centered at z and thus with boundary $S(z, r)$. Then, for some integer N, $\mathcal{M}$ is homologous to $NS(z, r)$ in $\mathbb{C}^2 \setminus \{z\}$, and, since $d_\zeta K_{BM}(\zeta, z) = 0$ in $\mathbb{C}^2 \setminus \{z\}$, we have, by Stokes' Theorem, that

$$\int_{\mathcal{M}} K_{BM}(\zeta, z) = N \int_{S(z,r)} K_{BM}(\zeta, z) = N.$$

This gives the claim. Since $I(z, \mathcal{M})$ is clearly continuous in z, we conclude that $I(z, \mathcal{M})$ is constant (and integer-valued) on each component U_j.

Finally, we claim that $I(z, \mathcal{M}) = 0$ for $z \in U_0$. It suffices to show this for large z. If z lies outside a large ball containing $\mathcal{M}$, then $\mathcal{M}$ is homologous to 0 inside that ball and the above application of Stokes' Theorem yields that $I(z, \mathcal{M}) = 0$.

We now take R such that $|w| > R$ for all $(z, w) \in M$ and define

$$\Phi(z, w) = \int_M \log(w - \eta)\pi^*(K_{BM}(\zeta, z)) \tag{16}$$

for $z \in U$ and $|w| > R$, where (ζ, η) gives a point of M. (This is completely analogous to the definition in (3), where the Cauchy integral is used for the one-dimensional case.) More precisely, the logarithm in (16) is defined, for a fixed w, only up to integer multiples of $2\pi i$. Hence, since the index is also an integer, Φ is well-defined up to integer multiples of $2\pi i$.

Thus we can put

$$F(z, w) = e^{\Phi(z,w)},$$

to get a single-valued function for $z \in U$ and $|w| > R$. For each i, $i = 0, 1, \ldots$, we define F_i as the restriction of F to $U_i \times \{|w| > R\}$.

In the previous case, analyticity of the Cauchy kernel in the z variable made it obvious that Φ is analytic. It the present case, since the Bochner–Martinelli kernel is not analytic in z, it is no longer obvious that Φ is analytic on $U \times \{|w| > R\}$. However, this is exactly where the hypothesis that M is maximally complex enters and yields the fact that indeed Φ is analytic. This follows from the following proposition.

We shall write K for the Bochner–Martinelli kernel K_{MB} in $\mathbb{C}^2$.

Proposition 19.7. *Let $\psi(\zeta, \eta)$ be a function analytic on a neighborhood of M in $\mathbb{C}^3$, where $\zeta = (\zeta_1, \zeta_2)$. Put*

$$F(z) = \int_M \psi(\zeta, \eta)\pi^*(K(\zeta, z)), \qquad z \in U.$$

Then F is analytic on U.

PROOF. We regard K as defined on $\mathbb{C}^2 \times \mathbb{C}^2 \setminus \{z = \zeta\}$. Here

$$K(\zeta, z) = \left(\frac{\bar{\zeta}_1 - \bar{z}_1}{|\zeta - z|^4} d\bar{\zeta}_2 - \frac{\bar{\zeta}_2 - \bar{z}_2}{|\zeta - z|^4} d\bar{\zeta}_1\right) \wedge d\zeta_1 \wedge d\zeta_2. \tag{17}$$

So $\bar{\partial}_z K(\zeta, z)$ is a form on $\mathbb{C}^2 \times \mathbb{C}^2 \setminus \{z = \zeta\}$ of type (2, 1) in ζ and type (0, 1) in z. We define a form K_1 on $\mathbb{C}^2 \times \mathbb{C}^2 \setminus \{z = \zeta\}$ that is of type (2, 0) in ζ and type (0, 1) in z by

$$K_1(\zeta, z) = \left(\frac{\bar{\zeta}_2 - \bar{z}_2}{|\zeta - z|^4} d\bar{z}_1 - \frac{\bar{\zeta}_1 - \bar{z}_1}{|\zeta - z|^4} d\bar{z}_2\right) \wedge d\zeta_1 \wedge d\zeta_2.$$

Then $\bar{\partial}_\zeta K_1(\zeta, z)$ is a form on $\mathbb{C}^2 \times \mathbb{C}^2 \setminus \{z = \zeta\}$ of type (2, 1) in ζ and type (0, 1) in z and so of the same type as $\bar{\partial}_z K(\zeta, z)$.

Lemma 19.8. $-\bar{\partial}_\zeta K_1(\zeta, z) = \bar{\partial}_z K(\zeta, z)$.

PROOF. See Appendix A13. □

Lemma 19.9. *For $(\zeta, \eta) \in M$ (and $z \in U$ fixed)*

$$\psi(\zeta, \eta)\pi^*(\bar{\partial}_\zeta K_1(\zeta, z)) = d(\psi(\zeta, \eta)\pi^*(K_1(\zeta, z))).$$

PROOF. Since the map π and the function ψ are holomorphic, we have

$$\begin{aligned} \psi(\zeta, \eta)\pi^*(\bar{\partial}_\zeta K_1(\zeta, z)) &= \bar{\partial}_\zeta(\psi(\zeta, \eta)\pi^*(K_1(\zeta, z))) \\ &= \bar{\partial}_{\zeta,\eta}(\psi(\zeta, \eta)\pi^*(K_1(\zeta, z))). \end{aligned}$$

This holds on the open set W of $\mathbb{C}^3$ containing M, where ψ is defined. On W we can write the exterior derivative d as the sum $d = \bar{\partial}_{\zeta,\eta} + \partial_{\zeta,\eta}$. We have $\partial_{\zeta,\eta} = \partial_\zeta + \partial_\eta$.

Since $\pi^*(K_1(\zeta, z))$ is of type (2, 0) in ζ, we have $\partial_\zeta(\psi(\zeta, \eta)\pi^*(K_1(\zeta, z))) = 0$. Hence we get on W:

$$\psi(\zeta, \eta)\pi^*(\bar{\partial}_\zeta K_1(\zeta, z)) = d(\psi(\zeta, \eta)\pi^*(K_1(\zeta, z))) - (\partial_\eta \psi) \wedge \pi^*(K_1(\zeta, z)).$$

Thus it remains to show that $\alpha = (\partial_\eta \psi) \wedge \pi^*(K_1(\zeta, z))$, a (3,0)-form in (ζ, η), is zero when restricted to M. This follows from the fact that M is maximally complex, as the next lemma shows. □

Lemma 19.10. *Let α be a form of type* $(3, 0)$ *defined on a neighborhood of a maximally complex real 3-manifold M in $\mathbb{C}^3$. Then the restriction of α to M is identically zero.*

Proof. We need only verify this at the tangent space $T_x(M)$ at a single point $x \in M$. By a linear change of variable we may assume, by the maximal complexity of M, that $T_x(M) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Im}(z_2) = 0, z_3 = 0\}$. We have $\alpha(x) = A dz_1 \wedge dz_2 \wedge dz_3$, since α is a $(3, 0)$-form. Then the restriction of $\alpha(x)$ to $T_x(M)$ vanishes since dz_3 vanishes on $T_x(M)$, and this gives Lemma 19.10.

By (17) and Lemma 19.8, then, we have

$$(18)\quad \bar{\partial}_z F(z) = \int_M \psi(\zeta, \eta) \bar{\partial}_z \pi^*(K(\zeta, z))$$
$$= \int_M \psi(\zeta, \eta) \pi^*(\bar{\partial}_z K(\zeta, z)) = -\int_M \psi(\zeta, \eta) \pi^*(\bar{\partial}_\zeta K_1(\zeta, z)).$$

By Lemma 19.9, we get

$$(19)\qquad \bar{\partial}_z F(z) = -\int_M d(\psi(\zeta, \eta) \pi^*(K_1(\zeta, z))) = 0,$$

by Stokes' Theorem. Thus F is analytic. □

For additional simplicity in the exposition we shall assume that π maps M one-one to $\mathcal{M}$ except over the self-intersection set Λ of $\mathcal{M}$, which we assume to be a compact subset of $\mathcal{M}$ of finite two-dimensional measure. Thus we can write M over $\mathcal{M} \setminus \Lambda$ as a graph $\eta = f(\zeta)$ for $\zeta \in \mathcal{M} \setminus \Lambda$. Consequently, we can also write

$$\Phi(z, w) = \int_{\mathcal{M}} \log(w - f(\zeta)) K_{BM}(\zeta, z)$$

for $z \in U$ and $|w| > R$.

We shall need to "cross over a boundary" between two adjacent components U_j, U_k. For this we use the analogue of Plemelj's theorem for the Bochner–Martinelli kernel. This can be formulated as follows. Let Ω^+ and Ω^- be two domains in $\mathbb{C}^2$ with common boundary set α, where α is a smooth three-dimensional manifold, oriented positively with respect to Ω^+. (About $\mathcal{M}$, this means that as z moves from one component Ω^- of U to another Ω^+ across $\mathcal{M}$, the index of $\mathcal{M}$ about z increases by 1.) Let g be a $\mathcal{C}^2$-smooth function defined on α. Put

$$G^+(z) = \int_\alpha g(\zeta) K_{BM}(\zeta, z), \qquad z \in \Omega^+$$

and

$$G^-(z) = \int_\alpha g(\zeta) K_{BM}(\zeta, z), \qquad z \in \Omega^-$$

See Appendix B of [HarL2] for the proof of the following generalization of Plemelj's formulae.

Jump Theorem. *G^+ and G^- have continuous extensions to α, again denoted G^+ and G^-. On α we have*

$$G^+(z) - G^-(z) = g(z), \qquad z \in \alpha.$$

We have chosen notation so that the analogue of Lemma 19.2 is valid in the present setup with the only change being that now the common boundary set α is a smooth 3-manifold instead of a real curve; otherwise, the statement and proof are the same. With the same change, Lemma 19.3 carries over.

To verify the analogue of Lemma 19.4 we need that

$$\Phi(z, w) = \int_M \log(w - \eta)\pi^*(K_{BM}(\zeta, z)) = 0$$

for large w and z. We have noted above that Φ is analytic on U_0. This means that, for w fixed with $|w|$ sufficiently large, $z \longmapsto \Phi(z, w)$ is analytic in $\mathbb{C}^2$ outside of some large ball. In particular, for z_2 fixed with $|z_2|$ sufficiently large, $\lambda \longmapsto \Phi(\lambda, z_2, w)$ is an entire function of $\lambda \in \mathbb{C}$. It is clear, by looking at the integral defining Φ, that $\Phi(\lambda, z_2, w) \to 0$ as $\lambda \to \infty$, for z_2, w fixed as above. By Liouville's theorem, we conclude that $\Phi(\lambda, z_2, w) = 0$ for z_2, w sufficiently large and for all λ. By analytic continuation it follows that $\Phi \equiv 0$ on U_0. This gives Lemma 19.4 in our setting.

From this point on we conclude the argument by repeating the proof of Theorem 19.1. As in Lemma 19.5, the F_j have meromorphic extensions to $U_j \times \mathbb{C}$. A holomorphic chain V_j is associated to each function F_j, and these V_j are pasted together over common boundaries of the U_j, U_k. We refer the reader to the work of Harvey and Lawson [HarL2] for the full details of the proof of this beautiful theorem. □

To conclude this chapter we shall sketch a single application of the general theorem of Harvey and Lawson. Let W be an analytic submanifold of $\mathbb{C}^3 \setminus K$ of complex dimension 2, where K is a compact subset of $\mathbb{C}^3$. Then W extends to be a subvariety of all of $\mathbb{C}^3$. To see this, we let M be the intersection of W with a sphere of large radius centered at the origin. Then M is maximally complex of real dimension 3. This is because *locally* M bounds the part W_1 of W lying outside of the sphere. By Theorem 19.6, M bounds a subvariety V inside the sphere. Finally, one can show that V and W_1 patch together across the sphere to give a global variety in $\mathbb{C}^3$.

20

Polynomial Hulls of Sets Over the Circle

I Introduction

Let X be a compact set in $\mathbb{C}^n$. Denote by π the projection $(z_1, \cdots, z_n) \mapsto z_1$ that maps $\mathbb{C}^n$ to the complex plane $\mathbb{C}$. $\pi(X)$ is a compact set in $\mathbb{C}$, and $\pi(\hat{X})$ is another such set. Of course,

$$(1) \qquad \pi(\hat{X}) \supseteq \pi(X),$$

and it can happen that

$$(2) \qquad \pi(\hat{X}) \neq \pi(X).$$

EXAMPLE 20.1. X = the torus $T^2 = \{(e^{i\theta_1}, e^{i\theta_2}) : \theta_1, \theta_2 \in \mathbb{R}\}$. Then $\hat{X}$ = the closed bidisk Δ^2, so $\pi(\hat{X}) = \Delta$ while $\pi(X)$ = the unit circle Γ.

It also can happen that even though $\hat{X}$ is strictly larger than X, we have

$$(3) \qquad \pi(\hat{X}) = \pi(X).$$

EXAMPLE 20.2. X = the sphere $\{|z_1|^2 + |z_2|^2 = 1\}$ in $\mathbb{C}^2$. Then $\hat{X}$ is the ball $\{|z_1|^2 + |z_2|^2 \leq 1\}$ and $\pi(\hat{X}) = \{|z_1| \leq 1\} = \pi(X)$.

If we are in the case of (2), we may consider a connected component W of the open set $\mathbb{C} \setminus \pi(X)$ with the property that there exists $z^0 = (z_1^0, \cdots, z_n^0) \in \hat{X}$ with $z_1^0 = \pi(z^0) \in W$. In that case,

$$(4) \qquad \pi(\hat{X}) \supseteq W.$$

Indeed, the following fact was verified in the first paragraph of the proof of Theorem 11.9.

Lemma 20.1. *Let $\mathfrak{A}$ be a uniform algebra on a compact space X. Let $f \in \mathfrak{A}$ and let W be a component of $\mathbb{C} \setminus f(X)$. If f takes a value λ_0 at a point of $\mathcal{M}$, where $\lambda_0 \in W$, then $f(\mathcal{M}) \supseteq W$.*

EXERCISE 20.1. Show that (4) follows from Lemma 20.1.

Suppose that X is a compact set in $\mathbb{C}^n$ such that (2) holds. Consider a component W of $\mathbb{C} \setminus \pi(X)$ such that $\pi(\hat{X}) \supseteq W$. If K is a compact subset of W, we put

$$\pi^{-1}(K) = \{z \in \hat{X} : \pi(z) \in K\}.$$

If $\pi^{-1}(K)$ is non-empty, then $\pi^{-1}(K)$ is a closed subset of $\hat{X}$ and π maps $\pi^{-1}(K)$ onto K, in view of (4). So we may think of $\pi^{-1}(K)$ as the portion of $\hat{X}$ that lies over K.

We fix a closed disk $\Delta \subseteq W$. Put $Y = \pi^{-1}(\partial\Delta)$.

Exercise 20.2. $\hat{Y} = \pi^{-1}(\Delta)$.

We use this fact as follows: If we can discover analytic structure in $\hat{Y}$, this provides us with analytic structure for that portion of $\hat{X}$ which lies over Δ. On the other hand, Y is a set lying over the circle $\partial\Delta$. One may hope that the fact that Y lies over a circle can be useful in the study of $\hat{Y}$, and this will turn out to be true.

We begin by taking $n = 2$ and Y a compact set in $\mathbb{C}^2$ lying over the unit circle $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. Clearly, to go from the unit circle to an arbitrary circle is a minor matter. For each $\lambda \in \Gamma$, we put

$$Y_\lambda = \{w \in \mathbb{C} : (\lambda, w) \in Y\}.$$

Y_λ is a compact set in the w-plane. We shall call it *the fiber over* λ. Strictly speaking, the fiber of the map π over λ is

$$\{(\lambda, w) : w \in Y_\lambda\}.$$

We denote by $\mathcal{F}$ the space of all functions f bounded and analytic on $\{|\lambda| < 1\}$ such that

(5) $$f(\lambda) \in Y_\lambda \text{ for a.a. } \lambda \in \Gamma.$$

Claim. Fix $f \in \mathcal{F}$. The graph of f,

$$\{(\lambda, f(\lambda)) : |\lambda| < 1\},$$

is contained in $\hat{Y}$.

Proof. Let P be a polynomial in λ and w. Then $g(\lambda) \equiv P(\lambda, f(\lambda)) \in H^\infty$. Also, for a.a. $\lambda \in \Gamma$, $f(\lambda) \in Y_\lambda$; therefore

(6) $$|P(\lambda, f(\lambda))| \leq ||P||_Y$$

for a.a. $\lambda \in \Gamma$. Hence

$$|g(\lambda_0)| \leq \operatorname{ess}\sup_\Gamma |g| \leq |P||_Y$$

for all λ_0 in the open unit disk, i.e.,

$$|P(\lambda_0, f(\lambda_0))| \leq ||P||_Y,$$

i.e, $(\lambda_0, f(\lambda_0)) \in \hat{Y}$. This gives the claim. $\square$

Theorem 20.2. *Fix a compact set Y in $\mathbb{C}^2$ lying over Γ. Assume*

$$Y_\lambda \text{ is convex for every } \lambda \in \Gamma. \tag{7}$$

Then $\hat{Y} \setminus Y$ equals the union of all graphs $\{(\lambda, f(\lambda)) : |\lambda| < 1\}$ with $f \in \mathcal{F}$.

PROOF. Because of the claim just proved, it suffices to show that if $(\lambda_0, w_0) \in \hat{Y} \setminus Y$, then there exists $f \in \mathcal{F}$ with $f(\lambda_0) = w_0$.

We first take $\lambda_0 = 0$. Since $(0, w_0) \in \hat{Y}$, we may choose a probability measure μ on Y such that, for each polynomial $P(\lambda, w)$,

$$P(0, w_0) = \int_Y P(\lambda, w) d\mu(\lambda, w). \tag{8}$$

Under the projection map $\pi : (\lambda, w) \mapsto \lambda$, μ "disintegrates" (see the Appendix) in the sense that there exists a probability measure μ_* on $\Gamma = \pi(Y)$, and for a.a. λ-dμ_* on Γ there exists a probability measure σ_λ on Y_λ, such that, for all $f \in C(Y)$,

$$\int_Y f d\mu = \int_\Gamma \left[\int_{Y_\lambda} f(\lambda, w) d\sigma_\lambda(w) \right] d\mu_*(\lambda).$$

In view of (8), we have for $n = 1, 2, \cdots, 0 = \int_Y \lambda^n d\mu = \int_\Gamma \lambda^n d\mu_*(\lambda)$.

This implies, writing $\lambda = e^{i\theta}$ for $\lambda \in \Gamma$, $\mu_* = \frac{1}{2\pi} d\theta$. (Why?) So we have for each $f \in C(Y)$

$$\int_Y f d\mu = \int_\Gamma \left[\int_{Y_\lambda} f(\lambda, w) d\sigma_\lambda(w) \right] \frac{1}{2\pi} d\theta. \tag{9}$$

We define

$$W(\lambda) = \int_{Y_\lambda} w d\sigma_\lambda, \qquad \lambda \in \Gamma.$$

W is defined a.a. on Γ and lies in $L^\infty(\Gamma, d\theta)$. For $n = 1, 2, \ldots,$

$$\int_\Gamma W(\lambda)\lambda^n \frac{d\theta}{2\pi} = \int_\Gamma \lambda^n \left[\int_{Y_\lambda} w d\sigma_\lambda \right] \frac{d\theta}{2\pi} = \int_Y \lambda^n w d\mu = 0,$$

because of (8) and (9).

It follows that $W \in H^\infty$. For $n = 0$, the same calculation yields

$$W(0) = \int_\Gamma W(\lambda) \frac{d\theta}{2\pi} = \int_Y w d\mu = w_0.$$

Thus W is the boundary value on Γ of a bounded analytic function on $\{|\lambda| < 1\}$ that takes the value w_0 at 0.

Finally, since σ_λ is a probability measure on Y_λ, for a.a. $\lambda \in \Gamma$, $\int_{Y_\lambda} w d\sigma_\lambda$ can be approximated arbitrarily closely by convex combinations of finite point-sets $w_1, \cdots, w_k$ in Y_λ. Since Y_λ is convex by hypothesis, and also is closed, $\int_{Y_\lambda} w d\sigma_\lambda \in Y_\lambda$. Thus $W(\lambda) \in Y_\lambda$ for a.a. λ in Γ. Thus $W \in \mathcal{F}$, and $W(0) = w_0$, as desired.

Next fix λ_0 in $\{|\lambda| < 1\}$. We shall reduce our problem to the previous case. We put

$$\chi(\lambda) = \frac{\lambda - \lambda_0}{1 - \bar{\lambda}_0 \lambda};$$

therefore, χ gives a homeomorphism of Γ onto Γ, and $\chi(\lambda_0) = 0$.

We define the set Y' over Γ by putting

$$Y'_\zeta = Y_{\chi^{-1}(\zeta)}, \qquad \zeta \in \Gamma.$$

Then Y' is compact. We verify that $(0, w_0) \in \hat{Y}'$. By the previous result, there exist $f \in H^\infty$, $f(0) = w_0$, and $f(\zeta) \in Y'_\zeta$ for a.a. $\zeta \in \Gamma$. Putting $g(\lambda) = f(\chi(\lambda))$, we then see that $g \in H^\infty$, $g(\lambda_0) = w_0$, and $g(\lambda) \in Y_\lambda$ for a.a. λ in Γ, as desired.

Finally, fix (λ_0, w_0) in $\hat{Y}$ with $\lambda_0 \in \Gamma$.

EXERCISE 20.3. $(\lambda_0, w_0) \in Y$.

This exercise completes the proof that $\hat{Y} \setminus Y = \{(\lambda, f(\lambda)) : |\lambda| < 1, f \in \mathcal{F}\}$. □

Consistent with our earlier notation, we now define, for $|\lambda| \le 1$,

$$\hat{Y}_\lambda = \{w \in \mathbb{C} : (\lambda, w) \in \hat{Y}\}.$$

Theorem 20.3. *Let Y be a compact set in $\mathbb{C}^2$ lying over Γ. Assume again that each fiber Y_λ, $\lambda \in \Gamma$, is convex. Then each fiber $\hat{Y}_\lambda$, $|\lambda| < 1$, is convex.*

PROOF. Fix λ_0, $|\lambda_0| < 1$, and choose points $w_1, w_2 \in \hat{Y}_{\lambda_0}$. By Theorem 20.2, there exist $f_1, f_2 \in \mathcal{F}$ with $f_j(\lambda_0) = w_j$, $j = 1, 2$. Put

$$f = \tfrac{1}{2}(f_1 + f_2).$$

Then f is bounded and analytic on $|\lambda| < 1$ and, for a.a. $\lambda \in \Gamma$, $f(\lambda) = \frac{1}{2}(f_1(\lambda) + f_2(\lambda)) \in Y_\lambda$, since $f_1(\lambda) \in Y_\lambda$, $f_2(\lambda) \in Y_\lambda$, and Y_λ is convex. Thus $f \in \mathcal{F}$. It follows that $(\lambda_0, f(\lambda_0)) \in \hat{Y}$. Thus $\frac{1}{2}(w_1 + w_2) = f(\lambda_0) \in \hat{Y}_{\lambda_0}$. So $\hat{Y}_{\lambda_0}$ is convex, as claimed. □

Theorems 20.2 and 20.3 suggest the following question: Given a family of convex sets Y_λ in the complex plane, defined for each $\lambda \in \Gamma$, such that the set

$$Y = \{(\lambda, w) \in \mathbb{C} : \lambda \in \Gamma, w \in Y_\lambda\}$$

is compact—for example, each fiber Y_λ, $\lambda \in \Gamma$, may be a line segment, a triangle, or an ellipse—how can we explicitly describe the family of compact sets $\hat{Y}_\lambda$, $|\lambda| < 1$? Or, equivalently, how can we explicitly describe the set $\hat{Y}$? A tractable example is given by the case where each Y_λ is a closed disk. In the next section, we shall study some examples of this case, and in the Notes we shall point out related, more general, results.

II Sets Over the Circle with Disk Fibers

We choose a continuous complex-valued function $\alpha\colon \lambda \mapsto \alpha(\lambda)$, defined on Γ, and we put

$$Y = \{(\lambda, w) : \lambda \in \Gamma, |w - \alpha(\lambda)| \leq 1\}.$$

Each Y_λ is then a disk of radius 1.

EXAMPLE 20.3. In the case $\alpha(\lambda) = \lambda$, and therefore also in the case $Y = \{(\lambda, w) : \lambda \in \Gamma, |w - \lambda| \leq 1\}$, we claim that $\hat{Y} = \{(\lambda, w) : |\lambda| \leq 1, |w - \lambda| \leq 1\}$.

To see this, fix $(\lambda_0, w_0) \in \hat{Y}$. Put $P(\lambda, w) = w - \lambda$. Then P is a polynomial and $|P| \leq 1$ on Y. Hence $|w_0 - \lambda_0| = |P(\lambda_0, w_0)| \leq 1$.

Conversely, fix (λ_0, w_0) with $|w_0 - \lambda_0| \leq 1$. Consider the analytic disk Σ, defined by $w - \lambda = w_0 - \lambda_0$, $|\lambda| \leq 1$. The boundary $\partial\Sigma$, lying over Γ, is contained in Y. If $Q(\lambda, w)$ is any polynomial, the restriction of Q to Σ is analytic and hence satisfies the maximum principle. Thus

$$|Q(\lambda_0, w_0)| \leq \max_{\partial\Sigma} |Q| \leq \max_{Y} |Q|.$$

So $(\lambda_0, w_0) \in \hat{Y}$, and therefore we are done.

□

EXERCISE 20.4. Let $\alpha(\lambda) = 2\bar{\lambda}$, $Y = \{(\lambda, w) : |w - 2\bar{\lambda}| \leq 1\}$. Show that $\hat{Y} \setminus Y$ is empty. [Hint: Apply Theorem 20.2.]

EXERCISE 20.5. Let $\alpha(\lambda) = \bar{\lambda}$, $Y = \{(\lambda, w) : |w - \bar{\lambda}| \leq 1\}$. Show that $\hat{Y} \setminus Y$ is the analytic disk $\{|\lambda| < 1, w = 0\}$.

A simple condition which assures that $\hat{Y} \setminus Y$ contains an open subset of $\mathbb{C}^2$ is the existence of a constant k such that $|\alpha(\lambda)| \leq k < 1$ for every λ in Γ. Suppose that this holds and put $Y = \{(\lambda, w) : |\lambda| = 1, |w - \alpha(\lambda)| \leq 1\}$. Fix r, $0 < r < 1 - k$. Then for each $\lambda \in \Gamma$, the disk $\{|w| \leq r\} \subseteq Y_\lambda$, and it follows that each "horizontal" disk: $w = w_0$, $|\lambda| \leq 1$, where $|w_0| \leq r$, is contained in $\hat{Y}$, and hence $\hat{Y}$ does have interior in $\mathbb{C}^2$.

We now consider the following special case. Let P, Q be polynomials in λ such that $Q \neq 0$ on Γ, and assume that there exists $k < 1$ such that

$$\left|\frac{P(\lambda)}{Q(\lambda)}\right| \leq k \qquad \forall \lambda \in \Gamma. \tag{10a}$$

Assume also that

(10*b*) Each zero of Q in $\{|\lambda| < 1\}$ is simple.

Put

$$Y = \left\{(\lambda, w) : \lambda \in \Gamma \text{ and } \left|w - \frac{P(\lambda)}{Q(\lambda)}\right| \leq 1\right\}.$$

In Theorem 20.5 below, we shall give an explicit description of $\widehat{Y_\lambda}$ for each λ in $\{|\lambda| < 1\}$.

We can write

$$Q(\lambda) = Q_0(\lambda)R(\lambda),$$

with

$$Q_0(\lambda) = \prod_{i=1}^{n} \frac{\lambda - \lambda_i}{1 - \bar{\lambda}_i \lambda},$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the zeros of Q in $\{|\lambda| < 1\}$ and R is a rational function with $R(\lambda) \neq 0$ in $\{|\lambda| \leq 1\}$. We put

$$w_j = -\frac{P(\lambda_j)}{R(\lambda_j)}, \qquad j = 1, 2, \cdots, n. \tag{11}$$

We let, as before, $\mathcal{F}$ denote the space of all functions $f \in H^\infty$ such that $f(\lambda) \in Y_\lambda$ a.e. on Γ.

Lemma 20.4. *$\mathcal{F}$ consists of all functions f on $\{|\lambda| < 1\}$ that can be written in the form*

$$f = \frac{P}{Q} + \frac{B}{Q_0} \tag{12}$$

with $B \in H^\infty$, $||B||_\infty \leq 1$, and

$$B(\lambda_j) = w_j, \qquad 1 \leq j \leq n. \tag{13}$$

Proof. Suppose that $f \in \mathcal{F}$. Define the function ζ by

$$f = \frac{P}{Q} + \zeta.$$

Then ζ is meromorphic on $\{|\lambda| < 1\}$ with a simple pole at each λ_j. Also,

$$\zeta = \frac{fQ - P}{Q} = \left(\frac{fQ - P}{R}\right)\frac{1}{Q_0}.$$

Put $B = (fQ - P)/R$. Then $B \in H^\infty$, $B(\lambda_j) = w_j$ for each j and, for a.a. $\lambda \in \Gamma$,

$$|B(\lambda)| = \left|\frac{fQ - P}{Q}(\lambda)\right| |Q_0(\lambda)| = \left|(f - \frac{P}{Q})(\lambda)\right| \leq 1.$$

Hence $||B||_\infty \leq 1$. Thus

$$f = \frac{P}{Q} + \frac{B}{Q_0},$$

where B satisfies (13) and $||B||_\infty \leq 1$.

Conversely, suppose that f has the representation (12) for some $B \in H^\infty$ with $||B||_\infty \leq 1$ and such that (13) holds. Then f is meromorphic on $\{|\lambda| < 1\}$ with (possibly removable) singularities at the λ_j and is regular elsewhere on $\{|\lambda| < 1\}$. We have

$$\operatorname{res}_{\lambda_j} \frac{P}{Q} = \frac{P(\lambda_j)}{Q'(\lambda_j)},$$

and

$$\operatorname{res}_{\lambda_j} \frac{B}{Q_0} = \frac{B(\lambda_j)}{Q_0'(\lambda_j)} = -\frac{P(\lambda_j)}{R(\lambda_j)} \frac{1}{Q_0'(\lambda_j)} = -\frac{P(\lambda_j)}{Q'(\lambda_j)},$$

$j = 1, \cdots, n$. Hence $\operatorname{res}_{\lambda_j} f = 0$ for all j. Since f has at each λ_j at most a simple pole, it follows that λ_j is a removable singularity for f for all j. So $f \in H^\infty$. Then, for almost all $\lambda \in \Gamma$,

$$\left| f(\lambda) - \frac{P(\lambda)}{Q(\lambda)} \right| = \left| \frac{B(\lambda)}{Q_0(\lambda)} \right| = |B(\lambda)| \le 1.$$

So $f \in \mathcal{F}$, and we are done.

To motivate what follows, we shall briefly review the interpolation theory in the disk developed by G. Pick and R. Nevanlinna early in the twentieth century.

Consider m distinct points $z_1, \ldots, z_m$ in $\{|z| < 1\}$. We ask for which m-tuples of complex numbers $\beta_1, \ldots, \beta_m$ there exists $F \in H^\infty$ such that

$$(14) \qquad ||F||_\infty \le 1 \text{ and } F(z_j) = \beta_j, \qquad 1 \le j \le m.$$

For each set $\beta_1, \cdots, \beta_m$ we denote by $M = M(z_1, \cdots, z_m | \beta_1, \cdots, \beta_m)$ the $m \times m$ Hermitian matrix

$$\left(\frac{1 - \beta_j \bar{\beta}_k}{1 - z_j \bar{z}_k} \right)_{j,k=1}^{m}.$$

Pick's Theorem. *Given $\beta_1, \cdots, \beta_m$ in $\mathbb{C}$, there exists $F \in H^\infty$ satisfying (14) if and only if the matrix M is positive semi-definite. Also, there exists $F \in H^\infty$ satisfying (14) and with $||F||_\infty < 1$ if and only if the matrix M is positive definite.*

We shall discuss this result in the Appendix.

Put $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| < 1 \text{ and } \lambda \neq \lambda_j \text{ for } j = 1, \cdots n\}$. We put, for $\lambda \in \Lambda$ and $w \in \mathbb{C}$,

$$(15) \qquad D(\lambda, w) = det(M(\lambda_1, \cdots, \lambda_n, \lambda | w_1, \cdots, w_n, w)),$$

where the w_j are given by (11). Note that D is real-valued since M is Hermitian.

EXAMPLE 20.4. Take $n = 2$. Then

$$D(\lambda, w) = \begin{vmatrix} \dfrac{1 - w_1\bar{w}_1}{1 - \lambda_1\bar{\lambda}_1} & \dfrac{1 - w_1\bar{w}_2}{1 - \lambda_1\bar{\lambda}_2} & \dfrac{1 - w_1\bar{w}}{1 - \lambda_1\bar{\lambda}} \\ \dfrac{1 - w_2\bar{w}_1}{1 - \lambda_2\bar{\lambda}_1} & \dfrac{1 - w_2\bar{w}_2}{1 - \lambda_2\bar{\lambda}_2} & \dfrac{1 - w_2\bar{w}}{1 - \lambda_2\bar{\lambda}} \\ \dfrac{1 - w\bar{w}_1}{1 - \lambda\bar{\lambda}_1} & \dfrac{1 - w\bar{w}_2}{1 - \lambda\bar{\lambda}_2} & \dfrac{1 - w\bar{w}}{1 - \lambda\bar{\lambda}} \end{vmatrix}.$$

EXERCISE 20.6. For each n and each $\lambda \in \Lambda$, $w \in \mathbb{C}$,

$$D(\lambda, w) = a|w|^2 + bw + \bar{b}\bar{w} + c,$$

where a, b, c are rational functions of $\lambda, \bar{\lambda}$.

Now fix polynomials P, Q as above, with $Q \neq 0$ on Γ, such that (10a) and (10b) hold. Define w_j by (11), $1 \leq j \leq n$.

Theorem 20.5. *For each $\lambda \in \Lambda$, we have*

(16) $\{w : D(\lambda, w) \geq 0\}$ *is a nondegenerate closed disk, and*

(17) $$\hat{Y}_\lambda = \left\{ \frac{P(\lambda)}{Q(\lambda)} + \frac{w}{Q_0(\lambda)} : D(\lambda, w) \geq 0 \right\}.$$

Remark. Formula (17) is the "explicit" description of $\hat{Y}$ we have been aiming at. Since the set $\{w : D(\lambda, w) \geq 0\}$ is a disk of positive radius, for each fixed $\lambda \in \Lambda$, (17) yields that $\hat{Y}_\lambda$ is a closed nondegenerate disk as well.

PROOF. We fix $\lambda \in \Lambda$ and put

$$E_\lambda = \{w : D(\lambda, w) > 0\}.$$

Claim. E_λ is nonempty.

PROOF. Fix $r < 1 - k$. As we saw earlier, our hypothesis that $|P/Q| \leq k$ on Γ implies that, for fixed w_0 with $|w_0| \leq r$, the disk $\{w = w_0, |\lambda| \leq 1\}$ lies in $\hat{Y}$, and so the constant function w_0 belongs to $\mathcal{F}$. By Lemma 20.4, this yields the existence of $B_0 \in H^\infty$ such that

(18) $$B_0(\lambda_j) = w_j, \qquad j = 1, \cdots, n,$$

and

(19) $$w_0 = \frac{P(\zeta)}{Q(\zeta)} + \frac{B_0(\zeta)}{Q_0(\zeta)}, \qquad |\zeta| < 1.$$

Then (19) holds a.e. on Γ. Hence we have

$$|B_0(\zeta)| = \left| w_0 - \frac{P(\zeta)}{Q(\zeta)} \right| \leq |w_0| + \left| \frac{P(\zeta)}{Q(\zeta)} \right| \leq r + k$$

for a.a. $\zeta \in \Gamma$. Hence $||B_0||_\infty \le r + k < 1$. Set $w' = (w_0 - P(\lambda)/Q(\lambda))Q_0(\lambda)$. We then have $B_0(\lambda_j) = w_j$, $1 \le j \le n$ and $B_0(\lambda) = w'$. By Pick's Theorem, it follows that

(20) $$M(\lambda_1, \cdots, \lambda_n, \lambda | w_1, \cdots, w_n, w') \text{ is positive definite,}$$

and hence $D(\lambda, w') > 0$. The claim is proved.

Further, (20) implies that

(21) $$M(\lambda_1, \cdots, \lambda_n | w_1, \cdots, w_n) \text{ is positive definite.}$$

Now fix w with $D(\lambda, w) > 0$. Then the matrix $M(\lambda_1, \cdots, \lambda_n, \lambda | w_1, \cdots, w_n, w)$ has all of its principal minors positive, because (20) shows this for all but the $(n+1) \times (n+1)$ minor, which equals $D(\lambda, w)$. It follows that $M(\lambda_1, \cdots, \lambda_n, \lambda | w_1, \cdots, w_n, w)$ is positive definite.

By Pick's Theorem, then, there exists $B \in H^\infty$, $||B||_\infty < 1$, with $B(\lambda_j) = w_j$, $1 \le j \le n$, and $B(\lambda) = w$. By Lemma 20.4, then

$$f = \frac{P}{Q} + \frac{B}{Q_0} \in \mathcal{F},$$

and so

$$\frac{P(\lambda)}{Q(\lambda)} + \frac{w}{Q_0(\lambda)} \in \hat{Y}_\lambda.$$

Assume next that $D(\lambda, w) \ge 0$. By the claim, $w = \lim_{n\to\infty} w_n$ with $D(\lambda, w_n) > 0$. Letting $n \to \infty$, we get that $\frac{P(\lambda)}{Q(\lambda)} + \frac{w}{Q_0(\lambda)} \in \hat{Y}_\lambda$. Thus the RHS (right-hand side) in (17) is contained in the LHS (left-hand side) in (17).

Finally, fix $w \in \hat{Y}_\lambda$. Then there exists $f \in \mathcal{F}$ with $f(\lambda) = w$ and so, by Lemma 20.4, there exists $B \in H^\infty$, $||B||_\infty \le 1$, $B(\lambda_j) = w_j$, $1 \le j \le n$ with $f = P/Q + B/Q_0$. We put $B(\lambda) = w'$. Then

$$w = f(\lambda) = \frac{P(\lambda)}{Q(\lambda)} + \frac{B(\lambda)}{Q_0(\lambda)} = \frac{P(\lambda)}{Q(\lambda)} + \frac{w'}{Q_0(\lambda)}.$$

Also, by Pick's Theorem, $M(\lambda_1, \cdots, \lambda_n, \lambda | w_1, \cdots, w_n, w')$ is positive semi-definite. Hence $D(\lambda, w') \ge 0$. So $w \in$ RHS in (17), and therefore LHS $\subseteq$ RHS.

Thus LHS = RHS and (17) holds. Also, the claim shows that $\{w : D(\lambda, w) \ge 0\}$ is nondegenerate, so (16) also holds. We are done. □

Theorem 20.5 leaves open the description of the fibers $\hat{Y}_\lambda$ for $\lambda \notin \Lambda$. The final result in this chapter fills this gap.

Corollary 20.6. *The fibers $\{\hat{Y}_\lambda : |\lambda| < 1\}$ form a continuously varying family of nondegenerate closed disks.*

Proof. We know that the fibers are nondegenerate since $\hat{Y}_\lambda \supseteq \{|w| \le 1 - k\}$ for all λ with $|\lambda| < 1$. Fix b in the open disk and suppose that $\{z_\nu\}$ is a sequence in Λ approaching b. By Theorem 20.5, the fiber for each z_ν is a closed disk D_ν. By the compactness of $\hat{Y}$, after passing to a subsequence, these disks converge to a nondegenerate disk $E \subseteq \hat{Y}_b$. Without loss of generality, we may assume that the disks converge for the original sequence.

Claim. $E = \hat{Y}_b$. Let $w \in \hat{Y}_b$. By Theorem 20.2, there exists $f \in \mathcal{F}$ such that $f(b) = w$. Also, $f(z_\nu) \in D_\nu$. Since $\lim_{\nu\to\infty} f(z_\nu) = f(b)$, we conclude that $w = f(b) \in E$. This shows that $E \supseteq \hat{Y}_b$ and gives the claim.

Thus we have that the fiber $\hat{Y}_b$ is a nondegenerate disk and that for every sequence $\{z_\nu\}$ in Λ there is a subsequence whose fibers converge to $\hat{Y}_b$. This yields the desired continuity of the fibers. $\square$

21

Areas

We begin with a theorem that gives a lower bound on the area of the spectrum of a member of a uniform algebra. Let $\mathcal{A}$ be a uniform algebra on the compact space X with maximal ideal space M. Let $\phi \in M$ and let μ be a representing measure supported on X for ϕ. We can view elements of $\mathcal{A}$ as continuous functions on M.

Theorem 21.1. *Let $f \in \mathcal{A}$ and suppose that $\phi(f) = 0$. Then*

$$\left[\pi \int |f|^2\, d\mu \leq area(f(M)).\right]$$

Applied to the disk algebra, this says that, for a function f in this algebra with $f(0) = 0$, we have $\frac{1}{2} \int |f|^2\, d\theta \leq \text{area}(f(D))$. Or, writing $f(z) = \sum_1^\infty a_n z^n$, we get $\pi \sum_1^\infty |a_n|^2 \leq \text{area}(f(D))$. This can be compared with the classical area formula $\int_D |f'|^2 dx\, dy = \text{area-with-multiplicity}(f(D))$, which gives $\pi \sum_1^\infty n|a_n|^2 = \text{area-with-multiplicity}(f(D))$.

For the proof of the theorem we need two lemmas. The first is an elegant computation due to Ahlfors and Beurling [AB]. The supremum of a function g over a compact set X is denoted by $||g||_X$.

Lemma 21.2. *Let K be a compact subset of the plane, and set*

$$F(z) = \int \int_K \frac{1}{\zeta - z}\, du dv, \qquad \textit{where } \zeta = u + iv.$$

Then F is continuous on the plane and $||F||_K \leq (\pi\ area(K))^{\frac{1}{2}}$.

Proof. The continuity of F is a consequence of the fact that the convolution of a local L^1 function with a bounded function of compact support is continuous. To estimate $F(z)$, we may assume, by a translation, that $z = 0$. Then, by a rotation we may assume that $F(0)$ is real and positive. Writing $\zeta = re^{i\theta}$, we have $F(0) = \text{Re}(F(0)) = \int \int_K \cos(\theta)\, dr\, d\theta$. Letting K^+ be the part of K in the right half-plane, we get $F(0) \leq \int \int_{K^+} \cos(\theta)\, dr\, d\theta$. Let $\ell(\theta)$ be the linear measure of the set of points ζ of K^+ with $\arg(\zeta) = \theta$. We have

$$F(0) \leq \int\int_{K^+} \cos(\theta) dr d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ell(\theta)\cos(\theta) d\theta \leq (\frac{\pi}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ell(\theta)^2 d\theta)^{\frac{1}{2}}.$$

The reader can verify that

$$\int_{\{r: re^{i\theta} \in K^+\}} r dr \geq \int_0^{\ell(\theta)} r dr = \frac{\ell(\theta)^2}{2}.$$

We conclude that

$$F(0) \leq (\pi \int\int_{K^+} r\, dr\, d\theta)^{\frac{1}{2}} = (\pi \text{ area}(K^+))^{\frac{1}{2}} \leq (\pi \text{ area}(K))^{\frac{1}{2}}.$$

□

We recall that $R(K)$ denotes the uniform closure in $C(K)$ of the rational functions with poles off of K. By an abuse of notion we write $\bar{z}$ as the conjugate function in the plane. We view $\bar{z}$ as an element of $C(K)$ in the following lemma.

Lemma 21.3. *dist*$(\bar{z}, R(K)) \leq (\frac{1}{\pi}\, \textit{area}(K))^{\frac{1}{2}}$.

Note that Lemma 21.3 is a quantitative version of the Hartogs–Rosenthal theorem that $R(K) = C(K)$ if K has zero area; for then $\bar{z}$ is in $C(K)$ and so $R(K) = C(K)$ by the Stone–Weierstrass theorem.

Proof of Lemma 21.3. Let ψ be a C^∞ function with compact support in the plane such that $\psi(z) \equiv \bar{z}$ on a neighborhood of K. By the generalized Cauchy integral formula,

$$\psi(z) = -\frac{1}{\pi} \int\int \frac{\partial\psi}{\partial\bar{\zeta}} \frac{dudv}{\zeta - z}, \qquad \zeta = u + iv,$$

for all $z \varepsilon \mathbb{C}$. Restricting ψ to K, and using $\frac{\partial\psi}{\partial\bar{\zeta}} \equiv 1$ on K we get

$$\bar{z} = -\frac{1}{\pi} \int\int_K \frac{dudv}{\zeta - z} - \frac{1}{\pi} \int\int_{K'} \frac{\partial\psi}{\partial\bar{\zeta}} \frac{dudv}{\zeta - z}$$

for $z \in K$, where $K' = \mathbb{C} \setminus K$. Since the integrand in the second integral is, for fixed ζ, a function in $R(K)$, it follows (approximate the integral by a sum!) that the second integral represents a function in $R(K)$. We get

$$\text{dist}(\bar{z}, R(K)) \leq \sup_{z \in K} |\frac{1}{\pi} \int\int_K \frac{dudv}{\zeta - z}|.$$

Lemma 21.3 follows by applying Lemma 21.2 to this last integral. □

Now we can prove Theorem 21.1. Let $\epsilon > 0$ and put $K = f(M)$. By Lemma 21.3 there is a rational function $r(z)$ with poles off of K such that

$$||\bar{z} - r(z)||_K < (\frac{\text{area}(K) + \epsilon}{\pi})^{\frac{1}{2}}.$$

Since r is holomorphic on a neighborhood of the spectrum K of f, it follows from the Gelfand theory that $g = r \circ f \in \mathcal{A}$ and $||\bar{f} - g||_M < ((\text{area}(K) + \epsilon)/\pi)^{\frac{1}{2}}$. We have $|f|^2 = f(\bar{f} - g) + fg$. Since $g \in \mathcal{A}$, we get $\int fg d\mu = \phi(f)\phi(g) = 0$ and so $\int |f|^2 \, d\mu = \int f(\bar{f} - g) d\mu$. Thus $\int |f|^2 \, d\mu \leq ||\bar{f} - g||_M \int |f| d\mu \leq ((\text{area}(K) + \epsilon)/\pi)^{\frac{1}{2}} \int |f| d\mu$. Now, letting $\epsilon \to 0$ yields

$$\int |f|^2 \, d\mu \leq (\frac{\text{area}(K)}{\pi})^{\frac{1}{2}} \int |f| d\mu.$$

Estimating the last integral by Hölder's inequality, $\int |f| d\mu \leq (\int |f|^2 d\mu)^{\frac{1}{2}}$, gives the theorem. □

As another application of Lemma 21.3 we shall give a proof of the classical isoperimetric inequality. This begins with a lower bound for dist$(\bar{z}, R(K))$.

Proposition 21.4. *Let Ω be a closed Jordan domain in the plane with a smooth boundary. Let A = area(Ω) and let L = length$(b\Omega)$. Then $2A/L \leq$ dist$(\bar{z}, R(\Omega))$.*

Proof. Let $\epsilon > 0$ and choose g a rational function with no poles on Ω such that $||\bar{z} - g||_\Omega <$ dist$(\bar{z}, R(\Omega)) + \epsilon$. We have, since $\int_{b\Omega} g dz = 0$ by Cauchy,

$$|\int_{b\Omega} \bar{z} dz| = |\int_{b\Omega} (\bar{z} - g) \, dz| < L(\text{dist}(\bar{z}, R(\Omega)) + \epsilon).$$

By Stokes' Theorem we have $2iA = \int_\Omega d\bar{z} dz = \int_{b\Omega} \bar{z} dz$. We get $2A < L(\text{dist}(\bar{z}, R(\Omega)) + \epsilon)$. Now the proposition follows by letting $\epsilon \to 0$.

Now, combining Lemma 21.3 and Proposition 21.4 gives

$$2A/L \leq \text{dist}(\bar{z}, R(\Omega)) \leq (A/\pi)^{1/2}.$$

The isoperimetric inequality follows directly:

$$4\pi A \leq L^2.$$

A different idea also can be used to study the metric properties of the set $f(M)$. Here f, M, X, $\mathcal{A}$, ϕ and μ are as in the first paragraph of this chapter. There exists a point $x_0 \in M$ such that $\phi(f) = f(x_0)$ for all $f \in \mathcal{A}$. (Recall that we view elements f of $\mathcal{A}$ as functions on M.) Let $\Gamma_t = \{z \in \mathbb{C} : |z| = t\}$. Let ℓ denote arclength measure on Γ_t.

Theorem 21.5. *Let $f \varepsilon \mathcal{A}$ and suppose that $\phi(f) = 0$. Then for all $t \geq 0$,*

$$2\pi t \mu\{x \in X : |f(x)| \geq t\} \leq \ell(\Gamma_t \cap f(M)).$$

Integrating this estimate from $t = 0$ to ∞ in fact implies Theorem 21.1. This follows from the general fact that

$$\int 2t\mu\{x \in X : |f(x)| \geq t\}dt = \int |f|^2 \, d\mu.$$

To verify this identity, we compute $\int 2t\chi(x,t) \; d\mu(x) \times dt$ (where χ is the characteristic function of the set $\{(x,t) \in X \times \mathbb{R}_+ : |f(x)| \geq t\}$) in two ways as iterated integrals, using Fubini's theorem.

PROOF OF THEOREM 21.5. Fix $t > 0$. Let $\epsilon > 0$ and put $\gamma = \Gamma_t \cap f(M)$. Choose a continuous real-valued function h on Γ_t such that $0 \leq h \leq 1$, h is identically 1 on a neighborhood of γ, and $\int_0^{2\pi} h(te^{i\theta})d\theta < t^{-1}(\ell(\gamma) + \epsilon)$. Let u be the harmonic extension of h to the interior of Γ_t, and let v be the harmonic conjugate of u with $v(0) = 0$. Set $F(z) = u(z) + iv(z)$ for $|z| < t$. Define $F(z)$ for $|z| > t$ by $F(z) = 2 - \overline{F(z^*)}$, where $z^* = t^2/\bar{z}$ is the reflection of z in Γ_t. Thus F is analytic off of Γ_t and, by the reflection principle, F is analytic on a neighborhood of γ. This means that F is analytic on $f(M)$.

By the Gelfand theory, $F \circ f \in \mathcal{A}$, and so

$$F(0) = F \circ f(x_0) = \int F \circ f \, d\mu.$$

We have

$$F(0) = u(0) = \frac{1}{2\pi} \int_0^{2\pi} h(te^{i\theta})d\theta < (2\pi t)^{-1}(\ell(\gamma) + \epsilon).$$

Taking real parts gives

$$\int \mathrm{Re}(F \circ f)d\mu < (2\pi t)^{-1}(\ell(\gamma) + \epsilon).$$

Since $0 < u(z) \leq 1$ on $|z| < t$, it follows that $1 \leq 2 - u(z)$. We conclude that $\mathrm{Re}F > 0$ on $f(M)$ and that $\mathrm{Re}F \geq 1$ on $\{z \in f(M) : |z| \geq t\}$. Hence

$$\int \mathrm{Re}(F \circ f)d\mu \geq \int_{\{x:|f(x)|\geq t\}} \mathrm{Re}(F \circ f)d\mu \geq \int_{\{x:|f(x)|\geq t\}} 1 \, d\mu$$
$$= \mu\{x : |f(x)| \geq t\}.$$

We now have $\mu\{x : |f(x)| \geq t\} \leq (2\pi t)^{-1}(\ell(\gamma) + \epsilon)$. Letting $\epsilon \to 0$ gives the theorem. □

We now shall apply Theorem 21.1 to analytic 1-varieties and polynomial hulls in $\mathbb{C}^n$.

Lemma 21.6. *Let V be a one-dimensional analytic subvariety of an open subset of $\mathbb{C}^n$. Then*

$$\mathit{area}(V) = \sum_{j=1}^{n} \mathit{area\text{-}with\text{-}multiplicity}(z_j(V)).$$

Here, by the area of V we understand the usual area of the set V_{reg} of regular points of V viewed as a two-dimensional real submanifold in $\mathbb{R}^{2n}$. This agrees with $\mathcal{H}^2(V)$, where $\mathcal{H}^2$ denotes two-dimensional Hausdorff measure in $\mathbb{C}^n$. See the Appendix for references to Hausdorff measure. The natural proof of this formula for varieties of arbitrary dimension k involves considering the form ω^k, where $\omega = i/2 \sum_j dz_j \wedge d\bar{z}_j$. We shall give a more classical proof in the case $k = 1$.

PROOF. This is a local result and so we can assume that V can be parameterized by a one-one analytic map $f : W \to V \subseteq \mathbb{C}^n$, where W is a domain in the complex plane. Let $\zeta \in W$ be $\zeta = s + it$ and let $f = (f_1, f_2, \ldots, f_n)$, where $f_k = u_k + iv_k$. Set $X(s,t) = f(\zeta)$ and view X as a map of W into $\mathbb{R}^{2n}$. The classical formula for the area of the map X is $\int_W |X_s||X_t| \sin(\theta)\, ds\, dt$, where θ is the angle between X_s and X_t. We have $X_s = (f_1', f_2', \ldots, f_n')$ and $X_t = (if_1', if_2', \ldots, if_n')$ by the Cauchy–Riemann equations. Hence $|X_t| = |X_s|$ and the vectors X_s and X_t are orthogonal in $\mathbb{R}^{2n}$, and so $\sin(\theta) \equiv 1$. Thus the previous formula for the area of the image of X becomes $\text{area}(V) = \int_W (\sum_j |f_j'|^2)\, ds\, dt = \sum_j \int_W |f_j'|^2\, ds\, dt$. Since $\int_W |f_j'|^2\, ds\, dt$ is the area-with-multiplicity of $f_j(W)$, this completes the proof. $\square$

Let V be a one-dimensional analytic subvariety of an open set Ω in $\mathbb{C}^n$. Let $p \in V$ and suppose that the closure of $B(p,r)$ (the open ball of radius r centered at p) is contained in Ω. Then it is a theorem of Rutishauser that the area of $V \cap B(p,r)$ is bounded below by πr^2. Our next result generalizes this by showing that the lower bound πr^2 for the "area of $V \cap B(p,r)$," which, by Lemma 21.6, equals the sum of the areas-with-multiplicity of the n coordinate projections, is in fact a lower bound for a smaller quantity, the "the sum of the areas (*without multiplicity*) of the coordinate projections." We shall prove this more generally for polynomial hulls. The connection between hulls and varieties is given by the following lemma.

Lemma 21.7. *Let V be a k-dimensional analytic subvariety of an open set Ω in $\mathbb{C}^n$. Suppose that the closure of the ball $B(p,r)$ is contained in Ω. Let $X = V \cap bB(p,r)$. Then $\hat{X} \cap B(p,r) = V \cap B(p,r)$.*

Now Rutishauer's result [Rut] is a consequence of the following fact about polynomial hulls.

Theorem 21.8. *Let X be a compact subset of $\mathbb{C}^n$. Suppose that $p \in \hat{X}$ and that $B(p,r) \subseteq \hat{X} \setminus X$. Then $\sum_{j=1}^n \mathcal{H}^2(z_j(\hat{X} \cap B(p,r))) \geq \pi r^2$.*

In general, polynomial hulls are not analytic sets, but the following shows that, locally, hulls that are not analytic sets have large area.

Theorem 21.9. *Let X be a compact subset of $\mathbb{C}^n$. Suppose that $p \in \hat{X} \setminus X$ and that, for some neighborhood N of p in $\mathbb{C}^n$, $\mathcal{H}^2(\hat{X} \cap N) < \infty$. Then $\hat{X} \cap N$ is a one-dimensional analytic subvariety of N.*

This yields another generalization of Rutishauser's result to polynomial hulls.

Corollary 21.10. *Let X be a compact subset of $\mathbb{C}^n$. Suppose that $p \in \hat{X}$ and that $B(p,r) \subseteq \mathbb{C}^n \setminus X$. Then $\mathcal{H}^2(\hat{X} \cap B(p,r)) \geq \pi r^2$.*

PROOF OF THE COROLLARY. If $\mathcal{H}^2(\hat{X} \cap B(p,r)) < \infty$, then the theorem implies that $\hat{X} \cap B(p,r)$ is a one-dimensional analytic set and so Rutishauser's theorem applies. If $\mathcal{H}^2(\hat{X} \cap B(p,r))$ is infinite, the conclusion is obvious. □

PROOF OF LEMMA 21.7. By the maximum principle, it follows that $V \cap B(p,r) \subseteq \hat{X}$.

Conversely, suppose that $q \in B(p,r) \setminus V$. Let $r' > r$ be such that $B(p,r') \subseteq \Omega$. There exists a function F that is holomorphic on $B(p,r')$ such that $F(q) = 1$ and $F \equiv 0$ on $V \cap B(p,r')$—this is by the solution to the second Cousin problem on $B(p,r')$. In particular, $F \equiv 0$ on X. Approximating F uniformly on $\bar{B}(p,r)$ by polynomials with the Taylor series, it follows that $q \notin \hat{X}$. This shows that $B(p,r) \setminus V \subseteq B(p,r) \setminus \hat{X}$. Hence we have the reverse inclusion $\hat{X} \cap B(p,r) \subseteq V \cap B(p,r)$. This gives the lemma. □

PROOF OF THEOREM 21.8. We may, without loss of generality, suppose that $p = 0$. Take $s < r$. Set $Z = X \cap bB(p,s)$. By the local maximum modulus theorem, it follows that $\hat{Z} = \hat{X} \cap \bar{B}(p,s)$. We let $\mathcal{A}$ be the uniform closure of the polynomials in $C(Z)$. Then the maximal ideal space M of $\mathcal{A}$ is just $\hat{Z} = \hat{X} \cap \bar{B}(p,s)$. Then, for $f \in \mathcal{A}$, $f \mapsto f(0)$ is a continuous homomorphism ϕ on $\mathcal{A}$ represented by a measure μ on Z. The coordinate function z_k belongs to $\mathcal{A}$ with $\phi(z_k) = 0$, and so by Theorem 21.1 we have $\pi \int |z_k|^2\, d\mu \leq \mathcal{H}^2(z_k(\hat{X} \cap \bar{B}(p,s))$. Now we sum over k and use the fact that $\sum_{k=1}^n |z_k|^2 \equiv s^2$ on Z to get $\pi s^2 \leq \sum_{k=1}^n \mathcal{H}^2(z_k(\hat{X} \cap \bar{B}(p,s))$. Letting s increase to r now gives the theorem. □

PROOF OF THEOREM 21.9. Without loss of generality we may suppose that $p = 0$. Consider complex hyperplanes though the origin. Since $\mathcal{H}^2(\hat{X} \cap N) < \infty$, there is a complex hyperplane H such that $\mathcal{H}^1(\hat{X} \cap N \cap H) = 0$. (See the Appendix on Hausdorff measure.) We may suppose that H is the hyperplane $\{z_n = 0\}$. We write $z = (z', z_n)$ for $z \in \mathbb{C}^n$ with $z' \in \mathbb{C}^{n-1}$. The fact that $\mathcal{H}^1(\hat{X} \cap N \cap H) = 0$ implies (Appendix) that $\hat{X} \cap N$ is disjoint from $\{(z', z_n) : ||z'|| = \delta, z_n = 0\}$ for almost all $\delta > 0$.

Fix $\delta > 0$ such that $B(0,\delta) \subset N$ and $\hat{X} \cap N$ is disjoint from $\{(z', z_n) : ||z'|| = \delta, z_n = 0\}$. Then there exists $\epsilon > 0$ such that $\hat{X} \cap N$ is disjoint from $\{(z', z_n) : ||z'|| = \delta, |z_n| \leq \epsilon\}$. Let $\Delta = \{(z', z_n) : ||z'|| < \delta$ and $|z_n| < \epsilon\}$ and set $\rho(z) = z_n$. We can assume, by choosing ϵ small enough, that $\bar{\Delta} \subseteq N$. Note that $(\hat{X} \cap N) \cap b\Delta$ is contained in the set where $\{|z_n| = \epsilon\}$. This is because $\hat{X} \cap N$ is disjoint from $\{(z', z_n) : ||z'|| = \delta, |z_n| \leq \epsilon\}$.

We restrict the map ρ to $\hat{X} \cap \Delta$ and consider the 4-tuple $(A, \hat{X} \cap \Delta, D(\epsilon), \rho)$, where A is the restriction of the polynomials to $\hat{X} \cap \Delta$ and $D(\epsilon) = \{\lambda \in \mathbb{C} : |\lambda| < \epsilon\}$. We claim that this is a maximum modulus algebra. This follows from the local maximum modulus principle and the fact that ρ is a proper map from $\hat{X} \cap \Delta$ to $D(\epsilon)$. The map is proper because $\hat{X} \cap b\Delta$ is contained in the set where $\{|z_n| = \epsilon\}$.

Since $\mathcal{H}^2(\hat{X} \cap \Delta) < \infty$, it follows (see the Appendix on Hausdorff measure) that, for almost all points λ of $D(\epsilon)$ with respect to planar measure, the set $\rho^{-1}(\lambda) \cap (\hat{X} \cap \Delta)$ is finite. Hence there exists a positive integer m and a measurable set E in $D(\epsilon)$ of positive planar measure such that $\rho^{-1}(\lambda) \cap (\hat{X} \cap \Delta)$ contains exactly m points for every $\lambda \in E$. We now can apply Theorem 11.8 to conclude (i) that $\#\rho^{-1}(\lambda) \leq m$ for every $\lambda \in D(\epsilon)$ and (ii) that $\hat{X} \cap \Delta$ has analytic structure. The analytic structure given by part (ii) of Theorem 11.8 yields (cf. Exercise 11.5) the fact that $\hat{X} \cap \Delta$ is a one-dimensional analytic set. □

NOTES

Rutishauser's work [Rut] appeared in 1950. Estimates on the volume of an analytic variety were an essential part of the important paper of E. Bishop [Bi4]. Theorems 21.1 and 21.5 appear in [Al7]. The version for the disk was given by Alexander, Taylor, and Ullman [ATU]. Applications of this to analytic varieties can be found in the book by Chirka [Chi1]. Gamelin and Khavinson [GK] discuss the connection between the isoperimetric inequality and the area theorem. Theorem 21.9 is due independently to Sibony [Sib] and Alexander [Al6].

22

Topology of Hulls

1

We begin by stating some basic results about Morse theory. A nice description of what we need about Morse theory is given in Part I of Milnor's text [Mi2]. We will restrict our attention to open subsets M of $\mathbb{R}^s$. Let ϕ be a smooth function on M. A point $p \in M$ is a critical point of ϕ if $\partial\phi/\partial x_k(p) = 0$ for $1 \leq k \leq s$. A critical point p is nondegenerate if the (real) Hessian matrix $(\partial^2\phi/\partial x_j \partial x_k(p))$ is nonsingular. Then one defines the *index* of ϕ at p to be the number of negative eigenvalues of this matrix. Nondegenerate critical points are necessarily isolated. A Morse function ρ on M is a smooth real function such that $M^a \equiv \{x \in M : \rho(x) \leq a\}$ is compact for all a, all critical points of ρ are nondegenerate, and $\rho(p_1) \neq \rho(p_2)$ for critical points $p_1 \neq p_2$. The following lemma produces Morse functions.

Morse's Lemma. *Let $f : M \to \mathbb{R}$ be a smooth function, let K be compact and U be open and relatively compact in M such that $K \subseteq U \subseteq M$, and let $s > 0$ be an integer. Then there exists a smooth function $g : M \to \mathbb{R}$ such that g has nondegenerate critical points on K, the derivatives of g uniformly approximate the corresponding derivatives of f up to order s on U, and g agrees with f off of U.*

The main results from Morse theory that we shall need are contained in the following theorem.

Morse's Theorem. *Let ρ be a Morse function on M.*

(a) *If ρ has no critical points in the set $\{x \in M : a \leq \rho(x) \leq b\}$, then M^a is diffeomorphic to M^b.*
(b) *If ρ has a single critical point p of index λ in the set $\{x \in M : a \leq \rho(x) \leq b\}$ and $a < \rho(p) < b$, then M^b has the homotopy type of M^a with a λ-cell attached.*

We will not give the definitions here. The reader may find it instructive for (b) to sketch level sets of the function $F(x) = \sum_{j=1}^{s-\lambda} x_j^2 - \sum_{j=s-\lambda+1}^{s} x_j^2$ near the origin in $\mathbb{R}^s$; the origin is a non-degenerate critical point of index λ for F. For our purposes, the important point is the following consequence of the theorem. Here, G is an arbitrary abelian group and, for spaces $Y \subseteq X$, $H_k(X, Y; G)$ is the relative kth homology group of (X, Y) with coefficients in G, the most important cases being $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{Z}_2$.

Corollary. *Let ρ be as in the theorem and $a < b$.*

(i) *In case (a), $H_k(M^b, M^a; G) = 0$ for all k.*

(ii) *In case (b), $H_\lambda(M^b, M^a; G) = G$ and $H_k(M^b, M^a; G) = 0$ for $k \neq \lambda$.*

(iii) *Suppose that the index of ρ is $\leq \sigma$ at every critical point of ρ. Then $H_k(M^b, M^a; G) = 0$ if $k \geq \sigma + 1$.*

(iv) *Suppose that the index of ρ is $\geq \sigma$ at every critical point of ρ. Then $H_k(M^b, M^a; G) = 0$ if $k \leq \sigma - 1$.*

We remark that the above theorem and corollary remain valid, with the same proof, if we relax the condition that $M^a \equiv \{x \in M : \rho(x) \leq a\}$ must be compact for all a in the definition of a Morse function to be that $\{x \in M : a \leq \rho(x) \leq b\}$ is compact whenever $a < b$.

2

Let ψ be a smooth strictly plurisubharmonic function on an open set Ω in $\mathbb{C}^n$.

Lemma 22.1. *Let p be an isolated nondegenerate critical point of ψ. Then the index of ψ at p is $\leq n$.*

Proof. We can assume that $p = 0$ and that $\psi(z) = \psi(0) + \sum c_{i\bar{j}} z_i \bar{z}_j + \operatorname{Re} \sum a_{ij} z_i z_j + O(|z|^3)$, where the first sum is positive definite, and so we can further assume that $c_{i\bar{j}} = \delta_{ij}$. Thus the real Hessian $2n \times 2n$ matrix of ψ at p is $I_{2n} + Q'$, where Q' is the matrix of the quadratic form on $\mathbb{R}^{2n}$ given by $X \to \operatorname{Re} \sum a_{ij} z_i z_j$, where $X = (\operatorname{Re} z, \operatorname{Im} z)$, $z = (z_1, z_2, \cdots, z_n)$. Since replacing z by iz in the quadratic form takes Q' to $-Q'$, and since the map $z \to iz$ induces an orthogonal map when viewed as a map of $\mathbb{R}^{2n}$, it follows that the matrices Q' and $-Q'$ are similar. Hence, if $v \in \mathbb{R}^{2n}$ is an eigenvalue of Q' of multiplicity m, then $-v$ is an eigenvalue of Q' of multiplicity m. Hence at least n of the eigenvalues of Q' are nonnegative. Therefore at least n of the eigenvalues of $I_{2n} + Q'$ are greater than or equal to 1. Thus the number of negative eigenvalues of $I_{2n} + Q'$ (the index) is at most n. □

Definition 21.1. A *Runge domain* Ω in $\mathbb{C}^n$ is an open subset of $\mathbb{C}^n$ such that $\hat{K} \subseteq \Omega$ for every compact subset K of Ω. (We remark that the definition here of Runge domain coincides with what is often called an *open polynomially convex set*. Our notion of Runge domain agrees with what is sometimes called a *holomorphically convex Runge domain*.)

The following two theorems are the main results that we shall present about the topology of polynomial hulls.

Theorem 22.2 (Andreotti and Narasimhan [AnNa]). *If Ω is Runge in $\mathbb{C}^n$, then*

$$H_k(\Omega; G) = 0 \qquad \text{for } k \geq n$$

Theorem 22.3 (Forstnerič [Fo2]). *Let K be a compact polynomially convex set in $\mathbb{C}^n$ $(n \geq 2)$. Then*

$$H_k(\mathbb{C}^n \setminus K; G) = 0 \qquad \text{for } 1 \leq k \leq n-1$$

and

$$\pi_k(\mathbb{C}^n \setminus K) = 0 \qquad \text{for } 1 \leq k \leq n-1.$$

Lemma 22.4. *Let K be polynomially convex in $\mathbb{C}^n$ with $K \subseteq U$, with U open and bounded. Then there exists a smooth strictly plurisubharmonic function $\rho : \mathbb{C}^n \to \mathbb{R}$ and $R > 0$ such that:*

(i) *$\rho < 0$ on K and $\rho > 0$ on $\mathbb{C}^n \setminus U$;*
(ii) *$\rho(z) = |z|^2$ for $|z| > R$; and*
(iii) *ρ is a Morse function on $\mathbb{C}^n$.*

Proof of the Lemma. We first construct a smooth strictly plurisubharmonic function v on $\mathbb{C}^n$ satisfying (i). Choose $C >$ the maximum of $|z|^2$ on K. Let $L = \{z \in \mathbb{C}^n : |z|^2 - C \leq 0\} \setminus U$. Then L is compact and disjoint from K. For all $x \in L$, we can choose a polynomial p_x such that $|p_x| < 1$ on K and $|p_x(x)| > 2$, and so $|p_x| > 2$ on a neighborhood of x. By the compactness of L we get a finite set of $\{p_x\}$ such that the maximum ϕ of their moduli satisfies $\phi > 2$ on L. Then ϕ is also plurisubharmonic and $\phi < 1$ on K. Now set $v_0 = \max(\phi - 1, |z|^2 - C)$. This is plurisubharmonic on $\mathbb{C}^n$ and satisfies the conditions of (i). Finally, by smoothing v_0 and adding $\epsilon|z|^2$ for small positive ϵ, we get the desired smooth strictly plurisubharmonic function v on $\mathbb{C}^n$, satisfying (i).

Choose $R > 0$ such that $\overline{U} \subseteq B(0, R/3)$. Let $h : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $h \geq 0$, $h \equiv 0$ for $t \leq R/3$, h is strictly convex and increasing for $t > R/3$, and $h(t) = t^2$ for $t \geq R$ (h can be constructed from its second derivative). Let χ be a smooth function on $\mathbb{R}$ such that $0 \leq \chi(t) \leq 1$, $\chi(t) = 1$ for $t < R/2$ and $\chi(t) = 0$ for $t \geq R$. Consider the smooth function

$$\rho(z) = h(|z|) + \epsilon\chi(|z|)v(z).$$

For sufficiently small $\epsilon > 0$, ρ is strictly plurisubharmonic and satisfies (i) and (ii). Since all of the critical points of ρ are contained in $B(0, R)$, we can, by the

Morse lemma, make a small modification of ρ on $B(0, R)$ so that it becomes a Morse function—the properties (i) and (ii) are preserved. $\square$

PROOF OF THEOREM 22.2. Let K be a compact subset of Ω. It suffices to show that $H_k(L; G) = 0$ for $k \geq n$ for some compact subset L of Ω such that $K \subseteq L$—this is because $H_k(\Omega; G)$ is the (inverse) limit of such $H_k(L; G)$. Replacing K by its polynomially convex hull, we may assume, since Ω is Runge, that K is polynomially convex. In Lemma 22.4, take U to be Ω and get a Morse function ρ on $\mathbb{C}^n$. Choose $a < 0$ a regular value of ρ such that $\rho < a$ on K. Then $M^a = \{z \in \mathbb{C}^n : \rho(z) \leq a\}$ is a compact subset of Ω and $K \subseteq M^a$. We take M^a to be the set L. For $a < b$ we have from Section 1, since the index of ρ is $\leq n$ at each critical point, that $H_k(M^b, M^a; G) = 0$ for $k \geq n + 1$. Letting $b \to \infty$ gives $H_k(\mathbb{C}^n, M^a; G) = 0$ for $k \geq n + 1$. From the long exact sequence we get

$$H_{k+1}(\mathbb{C}^n, M^a; G) \to H_k(M^a; G) \to H_k(\mathbb{C}^n; G) = 0$$

for $k \geq n$. We conclude that $H_k(M^a; G) = 0$. $\square$

PROOF OF THEOREM 22.3. Let U be an open set in $\mathbb{C}^n$ containing K. Apply the previous lemma to get ρ. Let $\psi = -\rho$. Then the critical points of ψ are nondegenerate and the index of ψ at each of its critical points p equals $2n$ (the index of ρ at p). Hence the index of ψ is $\geq n$ at each critical point. Now set $X^a \equiv \{x \in \mathbb{C}^n : \psi \leq a\}$. We apply the remark at the end of Section 1 to ψ. Using $-R^2 < 0$ in (iv) of the corollary of Section 1 gives $H_k(X^0, X^{-R^2}; G) = 0$ for $0 \leq k \leq n - 1$. From the long exact sequence we have

$$H_k(X^{-R^2}; G) \to H_k(X^0; G) \to H_k(X^0, X^{-R^2}; G)$$

for $0 \leq k \leq n - 1$. Note that $X^{-R^2} = \{z \in \mathbb{C}^n : |z| \geq R\}$ is topologically the product of an interval and S^{2n-1}. Hence $H_k(X^{-R^2}; G) = 0$ for $1 \leq k \leq n - 1$, and we conclude that $H_k(X^0; G) = 0$ for $1 \leq k \leq n - 1$. Note that $\mathbb{C}^n \setminus U \subseteq X^0 \subseteq \mathbb{C}^n \setminus K$ and hence that $H_k(\mathbb{C}^n \setminus K; G)$ is the (direct) limit of the $H_k(X^0; G)$ as U shrinks to K. We conclude that $H_k(\mathbb{C}^n \setminus K; G) = 0$ for $1 \leq k \leq n - 1$.

The second part of the theorem on homotopy groups follows by the same proof as just given for the homology groups. We omit the details. $\square$

From Theorem 22.2, we get a corresponding statement for the real singular cohomology groups of a Runge domain in $\mathbb{C}^n$: $H^k(\Omega; \mathbb{R}) = 0$ for $k \geq n$. This implies the same for the Čech cohomology groups (since these agree with the singular groups of open sets): $\check{H}^k(\Omega; \mathbb{R}) = 0$ for $k \geq n$.

Lemma 22.5. *If K in $\mathbb{C}^n$ is polynomially convex, then K is a decreasing limit of Runge domains. More precisely, if U is an open set containing K, then there exists a bounded Runge domain Ω such that*

$$K \subseteq \Omega \subseteq U.$$

Remark. We give the short proof; it is essentially the proof of Lemma 7.4. We could also appeal directly to Lemma 7.4, which, with a "change of scale," implies the present lemma.

PROOF. Choose a constant $C > 0$ such that $|z_j| \le C$ on K for $1 \le j \le n$ and set $L = \{z \in \mathbb{C}^n : |z_j| \le C \text{ for } 1 \le j \le n\} \setminus U$. Then L is compact and disjoint from K. For all $q \in L$, there exists a polynomial p_x such that $|p_x| < 1$ on K and $|p_x| > 2$ on a neighborhood of x. By the compactness of L there is a finite set of the $\{p_x\}$, call them $p_1, p_2, \cdots, p_s$, such that $|p_j| < 1$ on K and for all $x \in L$ there is a k such that $|p_k(x)| > 2$. Now put $\Omega = \{z : |z_j| < C \text{ for } 1 \le j \le n \text{ and } |p_j(z)| < 1 \text{ for } 1 \le j \le s\}$. ($\Omega$ is a "polynomial polyhedron"— this is slightly more general than the notion of p-polyhedron of Definition 7.3.) The reader can easily check that Ω is a Runge domain and that $K \subseteq \Omega \subseteq U$. $\square$

It follows from Lemma 22.5 by a basic continuity property of Čech cohomology that, for K polynomially convex, $\check{H}^k(K; \mathbb{R}) = 0$ for $k \ge n$. For "nice" sets K, the singular cohomology and Čech cohomology agree, and in those cases one can say that $H^k(K; \mathbb{R}) = 0$ for $k \ge n$ when K is polynomially convex. The same statements with coefficients $\mathbb{R}$ replaced by $\mathbb{C}$ are equally true.

Note that if $K \subseteq \mathbb{R}^n \subseteq \mathbb{C}^n$, then K is polynomially convex, by the Stone–Weierstrass approximation theorem. By choosing K to be a union of spheres of dimension $\le n-1$, one sees that the groups $H^k(K; \mathbb{R})$ for $0 \le k \le n-1$ need not vanish for polynomially convex sets.

We can now verify Browder's theorem (Theorem 15.8), that $\check{H}^n(\mathfrak{M}; \mathbb{C}) = 0$, where $\mathfrak{M}$ is the maximal ideal space of a Banach algebra $\mathfrak{A}$ generated by n elements. By the paragraph after the proof of Lemma 22.5 it suffices to show that $\mathfrak{M}$ is homeomorphic to a polynomially convex set in $\mathbb{C}^n$. This follows from Lemma 8.3 (semi-simplicity is not used in the argument of Lemma 8.3).

3

We now can obtain an application of Theorem 22.2. Let $\mathbb{B}_n$ denote the open unit ball in $\mathbb{C}^n$. Recall that $\hat{Y}$ denotes the polynomially convex hull of Y.

Theorem 22.6. *Let f be a continuous complex-valued function defined on $b\mathbb{B}_n$, $n \ge 2$. Let $G(f)$ be the graph of f in $\mathbb{C}^{n+1}$. Then $\widehat{G(f)}$ covers $\mathbb{B}_n$; i.e., the projection of the set $\widehat{G(f)}$ to $\mathbb{C}_n$ is $\overline{\mathbb{B}_n}$.*

Remark. This theorem is of course false for $n = 1$. In that case we know, by Chapter 20, that the part of $\widehat{G(f)}$ over the open unit disk is either empty or is an analytic graph. In particular, if f is real-valued and nonconstant, then $G(f)$ is polynomially convex.

PROOF. Let $\pi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection $\pi(z, z_{n+1}) = z$ for $z \in \mathbb{C}^n$. We argue by contradiction and suppose that $\pi(\widehat{G(f)})$ does not contain $\mathbb{B}_n$. By applying a biholomorphism of $\mathbb{B}_n$ we may assume (since the the group of biholomorphisms of the ball is transitive—see the Appendix) that $0 \notin \pi(\widehat{G(f)})$. Then $\widehat{G(f)} \subseteq (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}$. By Lemma 22.5, there exists a Runge domain Ω with $\widehat{G(f)} \subseteq \Omega \subseteq (\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}$. We can approximate f uniformly on $b\mathbb{B}_n$ by a smooth function g such that $G(g) \subseteq \Omega$.

Consider the Bochner–Martinelli form

$$\omega = \sum_{j=1}^{n} (-1)^{j-1} \frac{\overline{z}_j}{|z|^{2n}} d\overline{z}_1 \wedge \cdots \wedge \widehat{d\overline{z}_j} \wedge \cdots d\overline{z}_n \wedge dz_1 \cdots \wedge dz_n.$$

Then ω is a closed $(2n-1)$-form (i.e., $d\omega = 0$) on $\mathbb{C}^n \setminus \{0\}$. Let $\pi_0 : \Omega \to \mathbb{C}^n$ be the restriction of π to Ω. Let $\sigma = \pi_0^*(\omega)$ be the "pull back" of ω to Ω. Then, since $d\sigma = d\pi_0^*(\omega) = \pi_0^*(d\omega) = 0$, we have that σ is a closed $(2n-1)$-form on Ω. By Theorem 22.2, since Ω is Runge in $\mathbb{C}^{n+1}$ and $2n - 1 \geq n + 1$, $H_{2n-1}(\Omega; \mathbb{C}) = 0$ and therefore $H^{2n-1}(\Omega; \mathbb{C}) = 0$. Since the singular group $H^{2n-1}(\Omega; \mathbb{C})$ agrees with the deRham cohomology group, we conclude that σ is exact; i.e., there is a $2n-2$-form β on Ω such that $d\beta = \sigma$. Hence we get

$$\int_{bB_n} \omega = \int_{G(g)} \sigma = \int_{G(g)} d\beta = 0$$

by Stokes' Theorem, since $G(g) \subseteq \Omega$ is a smooth manifold without boundary. On the other hand, one has

$$\omega = \sum_{j=1}^{n} (-1)^{j-1} \overline{z}_j d\overline{z}_1 \wedge \cdots \wedge \widehat{d\overline{z}_j} \wedge \cdots d\overline{z}_n \wedge dz_1 \cdots \wedge dz_n$$

on $b\mathbb{B}_n$. Hence, by Stokes' Theorem,

$$\int_{bB_n} \omega = n \int_{B_n} d\overline{z}_1 \wedge \cdots d\overline{z}_n \wedge dz_1 \cdots \wedge dz_n \neq 0.$$

Contradiction. □

NOTES

The special case of Theorem 22.2 when $G = \mathbb{C}$ and $k > n$ can be proved for a pseudoconvex domain Ω via the complex DeRham Theorem; see the text of L. Hörmander [Hö2], Theorem 4.2.7. The case of Theorem 22.2 when $G = \mathbb{C}$ and $k = n$ is due to Serre [Ser]. The idea of using Morse theory appeared in the the papers of Andreotti and Frankel [AnFr] and Andreotti and Narasimhan [AnNa]. Morse theory, which yields more precise conclusions than we have stated here, has the advantage of giving results for arbitrary coefficient groups. Part I of Milnor's text [Mi2] gives a very readable introduction to the Morse theory required in this chapter.

The intuitive idea that the pair $(\hat{X}, X)$ is somehow like a manifold with boundary, where $\hat{X}$ is the polynomial hull of a compact set $X \subseteq \mathbb{C}^n$, can be made precise with the concept of the linking number of two manifolds in $\mathbb{C}^n$. Connections between polynomial hulls and linking were given in [Al4]; the proofs use Theorem 22.2. This was further developed by Forstnerič in [Fo2], which contains Theorem 22.3.

Additional applications of Theorem 22.2 to hulls were given in [Al3]. In particular, Theorem 22.6 appeared in that paper with two proofs; the one that was suggested by J. P. Rosay appears here.

23

Pseudoconvex sets in $\mathbb{C}^n$

1 Smoothly Bounded Domains

Consider a region Ω in $\mathbb{R}^N$, given by a condition

$$\rho(x) < 0,$$

where ρ is a $\mathcal{C}^2$-function defined in a neighborhood of $\bar{\Omega}$, such that

$$\nabla\rho = (\frac{\partial\rho}{\partial x_1}, \cdots, \frac{\partial\rho}{\partial x_N}) \neq 0 \qquad \text{on } \partial\Omega.$$

Fix a point $x^0 \in \partial\Omega$ and put

$$(1) \qquad T_{x^0} = \{\xi \in \mathbb{R}^N : \sum_{j=1}^{N} \left(\frac{\partial\rho}{\partial x_j}\right)_0 \xi_j = 0\}$$

where the notation $(\cdot)_0$ means "evaluated at x^0."

T_{x^0} is the tangent hyperplane to $\partial\Omega$ at x^0. We can tell whether or not Ω is convex by looking at these T_{x^0}, $x^0 \in \Omega$. Ω is convex if and only if the translate of T_{x^0}, $\{\xi + x^0 : \xi \in T_{x^0}\}$, lies entirely outside of Ω, and this is equivalent to

$$(2) \qquad \sum_{j,k=1}^{N} \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}\right)_0 \xi_j\xi_k \geq 0 \qquad \forall\xi \in T_{x^0}.$$

In the early twentieth century, E.E. Levi discovered that a complex analogue of condition (2), in $\mathbb{C}^n$, has significance in the theory of analytic functions of n complex variables. We now consider a domain Ω in $\mathbb{C}^n$, defined by an inequality

$$\rho < 0,$$

with ρ as above. Note that since ρ is a real-valued function, the condition $\nabla\rho \neq 0$ is equivalent to

$$(\frac{\partial\rho}{\partial z_1}, \cdots, \frac{\partial\rho}{\partial z_n}) \neq 0.$$

We replace the differential operators $\frac{\partial}{\partial x_j}$ and $\frac{\partial^2}{\partial x_j \partial x_k}$ on $\mathbb{R}^N$ by the operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial^2}{\partial z_j \partial \bar{z}_k}$ on $\mathbb{C}^n$. This leads us to define a *complex tangent vector* to $\partial\Omega$ at z^0 to be a vector $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{C}^n$ satisfying

$$(1') \qquad \sum_{j=1}^{n} \left(\frac{\partial \rho}{\partial z_j} \right)_0 \zeta_j = 0.$$

EXERCISE 23.1. Fix $z_0 \in \partial\Omega$ and fix a vector $\zeta \in \mathbb{C}^n$. Let L denote the complex line consisting of all vectors $w\zeta$, $w \in \mathbb{C}$. Show that L is contained in the real tangent space T_{z^0} to $\partial\Omega$ at z_0 if and only if ζ satisfies (1′).

Fix $z^0 \in \Omega$.

Definition 23.1. $\partial\Omega$ is *pseudoconvex in the sense of Levi at* z^0 if ρ satisfies

$$(2') \qquad \sum_{j,k=1}^{n} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right)_0 \zeta_j \bar{\zeta}_k \geq 0$$

$$\forall \zeta \text{ in the complex tangent space to } \partial\Omega \text{ at } z^0.$$

Thus pseudoconvexity of $\partial\Omega$, in the sense of Levi, is the analogue of convexity in $\mathbb{R}^N$ if we replace the real tangent vectors ξ in $\mathbb{R}^N$ by the complex tangent vectors ζ in $\mathbb{C}^n$ and replace the symmetric quadratic form in (2) by the Hermitinan form in (2′).

EXAMPLE 23.1. Let a, b be positive constants. Put $\Omega = \{(z_1, z_2) : a|z_1|^2 + b|z_2|^2 < 1\}$. Fix $z^0 = (z_1^0, z_2^0) \in \partial\Omega$. Then $\zeta = (\zeta_1, \zeta_2)$ is a complex tangent vector to $\partial\Omega$ at z^0 if and only if $a\bar{z}_1\zeta_1 + b\bar{z}_2\zeta_2 = 0$. We have

$$\sum_{j,k=1}^{n} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right)_0 \zeta_j \bar{\zeta}_k = a|\zeta_1|^2 + b|\zeta_2|^2.$$

Hence (2′) is satisfied and so $\partial\Omega$ is pseudoconvex at z^0. (*Note:* In this case, (2′) is evidently satisfied for every $\zeta \in \mathbb{C}^2$. In general, at a pseudoconvex point, this will not be so.)

EXERCISE 23.2.
(a) Let Φ be an entire function on $\mathbb{C}^2$ such that $d\Phi \neq 0$ on $|\Phi| = 1$. Put $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |\Phi(z_1, z_2)| \leq 1\}$. Show, by direct computation, that $\partial\Omega$ is pseudoconvex at each point.
(b) Show that pseudoconvexity is invariant under local biholomorphisms.
(c) Use (b) to give an alternate proof of (a).

EXERCISE 23.3. Show that if Ω is the exterior of the unit ball in $\mathbb{C}^2$, i.e.,

$$\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 > 1\},$$

then $\partial\Omega$ is not pseudoconvex at any point.

2 Exhaustion Functions

We wish to extend the notion of pseudoconvexity to domains with nonsmooth boundaries. To point the way toward a good definition, we first look at a convex domain Ω in $\mathbb{R}^N$ without making any smoothness assumption on $\partial\Omega$. By an *exhaustion function for* Ω we mean a continuous real-valued function U defined on Ω such that the sublevel sets

$$K_t = \{x \in \Omega : U(x) \leq t\}, \quad t \in \mathbb{R},$$

are compact.

Since Ω is convex, Ω possesses an exhaustion function that is a convex function. Now let Ω be a domain in $\mathbb{C}^n$. The complex analogue of a convex exhaustion function is a plurisubharmonic function (see the Appendix) on Ω, which is an exhaustion function for Ω.

Definition 23.2. Let Ω be a domain in $\mathbb{C}^n$. Ω is *pseudoconvex* if there exists a continuous plurisubharmonic exhaustion function on Ω.

If Ω is pseudoconvex, then $\partial\Omega$ is pseudoconvex in the sense of Levi at each smooth point, and Definitions 23.1 and 23.2 are consistent in the sense that a smoothly bounded domain $\Omega \subseteq \mathbb{C}^n$ is pseudoconvex in the sense of Definition 23.2 if and only if $\partial\Omega$ is pseudoconvex in the sense of Levi at each point.

We have the following result. Let us put, for $z \in \Omega$,

$$\delta(z) = \text{distance}(z, \partial\Omega).$$

Proposition. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then Ω is pseudoconvex if and only if the function*

$$z \longmapsto -\log \delta(z)$$

is plurisubharmonic on Ω.

PROOF. See [Hö2], Chapter II.

We note that since $\delta(z) \to 0$ as $z \to \partial\Omega$, $-\log\delta(z)$ is an exhaustion function for Ω. □

3 Pseudoconvexity and Polynomial Hulls in $\mathbb{C}^2$

In this section we shall show that polynomial hulls in $\mathbb{C}^2$ have a close relationship with pseudoconvex domains. We shall prove the following result of Slodkowski [Sl1].

Theorem 23.1. *Fix a compact set Y in $\mathbb{C}^2$ and fix a point $p^0 \in \hat{Y} \setminus Y$. Let B be an open ball in $\mathbb{C}^2$, centered at p^0, such that $\bar{B}$ does not meet Y. Then each connected component of $B \setminus \hat{Y}$ is pseudoconvex.*

Why should we expect this theorem to be true? Let Y be a smooth closed curve in $\mathbb{C}^2$ with $\hat{Y} \setminus Y$ nonempty. Then $\hat{Y} \setminus Y$ is a one-dimensional analytic variety by Chapter 12. Fix $p^0 \in \hat{Y} \setminus Y$. In a small ball B centered at p^0, $\hat{Y}$ is defined by an equation,

$$F(z_1, z_2) = 0,$$

where F is analytic in $\bar{B}$. We put $\psi(z_1, z_2) = -\log|F(z_1, z_2)| - \log\delta(z_1, z_2)$, where δ is the distance function to the boundary of B. By the proposition above, $-\log\delta(z_1, z_2)$ is plurisubharmonic on B. Also, $\log|F|$ is pluriharmonic on $B \setminus \hat{Y}$.

Fix $z' \in \partial(B \setminus \hat{Y})$ and let $z^{(n)}$ be a sequence of points in $B \setminus \hat{Y}$ converging to z'. Either $z' \in \partial B$ or $z' \in \hat{Y}$. In the first case $-\log(\delta(z^{(n)})) \to +\infty$, and in the second case $-\log|F(z^{(n)})| \to +\infty$. In either case $\psi(z^{(n)}) \to +\infty$. It follows that the sublevel sets $\{\psi \le c\}$ are compact. So ψ is a plurisubharmonic exhaustion function of $B \setminus \hat{Y}$, and so $B \setminus \hat{Y}$ is pseudoconvex.

EXERCISE 23.4. Let B be a ball in $\mathbb{C}^2$ and let δ denote the distance function to ∂B. Show by direct calculation that $-\log\delta$ is plurisubharmonic in B.

EXERCISE 23.5. Verify Theorem 23.1 by direct calculation for the case that Y is the torus: $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$.

To prove the theorem we shall relate pseudoconvexity in $\mathbb{C}^2$ to "Hartogs figures" in $\mathbb{C}^2$.

Let $P = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| \le 1, |z_2| \le 1\}$, the unit polydisk in $\mathbb{C}^2$, and let $0 < q_1, q_2 < 1$. Set $H = \{z = (z_1, z_2) \in P : |z_1| \ge q_1 \text{ or } |z_2| \le q_2\}$. Then (P, H) is a *Euclidean Hartogs figure* in $\mathbb{C}^2$. Suppose that Φ is a biholomorphism $\Phi : P \to \mathbb{C}^2$. Set $\tilde{P} = \Phi(P)$ and $\tilde{H} = \Phi(H)$. Then $(\tilde{P}, \tilde{H})$ is a *general Hartogs figure* in $\mathbb{C}^2$. See [GF] for the generalization to $\mathbb{C}^n$.

Remark. Given a general Hartogs figure $(\tilde{P}, \tilde{H})$ in $\mathbb{C}^2$ with $\tilde{P} = \Phi(P)$. Consider the analytic disks $F_w : D \to \mathbb{C}^2$, where D is the closed unit disk, given by $F_w(\lambda) = \Phi(\lambda, w)$. Suppose that $\tilde{H} \subseteq \Omega$ and $\tilde{P} \not\subseteq \Omega$. Then, for all w, the boundary of F_w is contained in the compact subset $\tilde{H}$ of Ω. However, since $\tilde{P} \not\subseteq \Omega$, for some w, the disk $F_w(D)$ is not contained in Ω. But for $w = 0$, $F_0(D)$ is contained in $\tilde{H} \subseteq \Omega$. In other words, we have a continuous one (complex) parameter family $\{F_w\}$ of analytic disks all of whose boundaries lie in a fixed compact subset of Ω, such that $F_0(D)$ is contained in Ω but such that F_w is not contained in Ω for some w.

Lemma 23.2. *Let W be a bounded domain in $\mathbb{C}^2$ that is not pseudoconvex. Then exists a general Hartogs figure $(\tilde{P}, \tilde{H})$ in $\mathbb{C}^2$ such that $\tilde{H} \subseteq W$ and $\tilde{P} \not\subseteq W$.*

Remark. If $W \subseteq B$, where B is a ball, then $\tilde{P} \subseteq B$. Indeed, this follows by applying the maximum principle to the function $\rho(z, w) = |z|^2 + |w|^2$ to each of the disks $F_w(D)$, the boundary of which is contained in B. Alternatively, $F_w(D) \subseteq \widehat{F_w(\partial D)} \subseteq B$ since $F_w(\partial D) \subseteq B$.

PROOF OF LEMMA 23.2. (Cf. [Hö2], proof of Theorem 2.6.7) We assume that W is not pseudoconvex. Let $\delta(z)$ be the distance from z to ∂W. Then $-\log \delta$ is not plurisubharmonic in W. Therefore there exists a complex line L and there exists a disk $D_0 \subseteq W \cap L$ such that $-\log \delta$, restricted to L, violates the mean value inequality on D_0. We write $D_0 : z = z^0 + \tau\omega$, $|\tau| \leq r$, where z^0 and ω are fixed vectors in $\mathbb{C}^2$. Hence there exists a polynomial f, nonconstant, such that

$$(3) \qquad -\log \delta(z^0 + \tau\omega) \leq \operatorname{Re} f(\tau), \quad |\tau| = r$$

and

$$(4) \qquad -\log \delta(z^0) > \operatorname{Re} f(0).$$

Indeed, we first obtain (3) and (4) for $\operatorname{Re}(f)$ replaced by a harmonic function $h(\tau)$ and then approximate h by the real part of a polynomial f. We may assume that $f(0)$ is real. By (4),

$$\delta(z^0) < e^{-\operatorname{Re} f(0)} = e^{-f(0)}.$$

We choose $\epsilon > 0$ with

$$(5) \qquad \delta(z^0) < (1 - \epsilon)e^{-f(0)}.$$

Fix $z' \in \partial W$ such that $|z' - z^0| = \delta(z^0)$. Then put $a = (1 - \epsilon)\frac{z'-z^0}{|z'-z^0|}$, so $|a| = 1 - \epsilon$. Finally, put

$$(6) \qquad \tilde{z} = z^0 + e^{-f(0)}a.$$

We claim that a and ω are linearly independent over $\mathbb{C}$. To see this, we argue by contradiction. If not, then the disk D_0 and the point z' both lie in the complex line L. Say that $z' = z^0 + \tau'\omega$ for some τ' with $|\tau'| > r$. It follows that (i) $\delta(z^0 + \tau\omega) \leq |\tau - \tau'||\omega|$ for $|\tau| = r$ and (ii) $\delta(z^0) = |\tau'||\omega|$. Now we define a harmonic function h on $|\tau| \leq r$ by

$$h(\tau) = \operatorname{Re}(f(\tau)) + \log|\tau - \tau'| + \log|\omega|.$$

Then by (i) and (3) we have $h \geq 0$ on $|\tau| = r$. On the other hand, by (ii) and(4), we have $h(0) < 0$. Contradiction! The claim follows.

By (6) and (5),

$$|\tilde{z} - z^0| = |e^{-f(0)}||a| = e^{-f(0)}(1 - \epsilon) > \delta(z^0).$$

Thus $\tilde{z}$ is further along the ray from z^0 to z' than z' itself, and so the segment from z^0 to $\tilde{z}$ contains z'.

We next define for each real λ, $0 \leq \lambda \leq 1$,

$$z_\lambda(\tau) = z^0 + \tau\omega + \lambda e^{-f(\tau)}a,$$

$|\tau| \le r$, and $D_\lambda = \{z_\lambda(\tau) : |\tau| \le r\}$.

For each λ, $0 \le \lambda \le 1$,

$$z_\lambda(0) = z^0 + \lambda e^{-f(0)}a.$$

Hence for $\lambda = 0$, $z_\lambda(0) = z^0$ and for $\lambda = 1$, $z_\lambda(0) = z^0 + e^{-f(0)}a = \tilde{z}$. Since $z_\lambda(0)$ moves continuously with λ along the segment from z^0 to $\tilde{z}$, there exists λ_0 with $z_{\lambda_0}(0) = z'$, and hence D_{λ_0} contains z'.

We denote by λ_1 the smallest λ such that D_λ meets ∂W. Then

$$0 < \lambda_1 \le \lambda_0 \le 1.$$

Consider the map

$$\Phi(\tau, \lambda) = z^0 + \tau\omega + \lambda e^{-f(\tau)}a$$

when (τ, λ) varies over a neighborhood of $\{|\tau| \le r\} \times \{\lambda_1\}$ in $\mathbb{C}^2$. Note that, for all λ, D_λ is the image of $\{|\tau| \le r\} \times \{\lambda\}$ under Φ.

The determinant of the Jacobian matrix of Φ is given by

$$\begin{vmatrix} \omega_1 - \lambda f'(\tau)e^{-f(\tau)}a_1 & e^{-f(\tau)}a_1 \\ \omega_2 - \lambda f'(\tau)e^{-f(\tau)}a_2 & e^{-f(\tau)}a_2 \end{vmatrix} = \begin{vmatrix} \omega_1 & e^{-f(\tau)}a_1 \\ \omega_2 & e^{-f(\tau)}a_2 \end{vmatrix} = e^{-f(\tau)} \begin{vmatrix} \omega_1 & a_1 \\ \omega_2 & a_2 \end{vmatrix} \neq 0,$$

where $a = (a_1, a_2)^T$ and $\omega = (\omega_1, \omega_2)^T$, because a and ω are linearly independent. Moreover, it is straightforward to check that Φ is one-one on the disk $\{|\tau| \le r\} \times \{\lambda = \lambda_1\}$. It follows that if Δ is a sufficiently small closed disk centered at λ_1, then Φ is a biholomorphism of $\{|\tau| \le r\} \times \Delta$ to $\mathbb{C}^2$.

We claim that Φ yields the required general Hartogs figure. Indeed, if $z \in \partial D_{\lambda_1}$, then $|z - (z^0 + \tau\omega)| = |\lambda_1||e^{f(\tau)}||a| \le (1 - \epsilon)e^{-\mathrm{Re} f(\tau)}$. By (3), $\delta(z^0 + \tau\omega) \ge e^{-\mathrm{Re} f(\tau)}$. Therefore,

$$|z - (z^0 + \tau\omega)| \le (1 - \epsilon)\delta(z^0 + \tau\omega). \tag{7}$$

Hence $\partial D_{\lambda_1} \subseteq W$ and so, if $T \subseteq W$ is a compact neighborhood of ∂D_{λ_1}, then $\partial D_\lambda \subseteq T$ if λ lies in Δ and the radius of Δ is sufficiently small. Let Δ' be a closed subdisk of Δ centered at $\lambda' < \lambda_1$ such that $\lambda_1 \in \Delta'$. Then by choice of λ_1 we have $D_{\lambda'} \subseteq W$ and $D_{\lambda_1} \not\subseteq W$.

Now $\Phi : \{|\tau| \le r\} \times \Delta' \to \mathbb{C}^2$ is a biholomorphism that gives the required general Hartogs figure of Lemma 23.2, except for the fact that the domain of Φ is the polydisk $\{|\tau| \le r\} \times \Delta'$ rather than the unit polydisk. Composition with a simple affine change of variables that takes $(0, \lambda')$ to $(0, 0) \in \mathbb{C}^2$ yields a Φ that satisfies this final requirement. □

Proof of Theorem 23.1. Let Ω be a connected component of $B \setminus \hat{Y}$. Arguing by contradiction, we suppose that Ω is not pseudoconvex. Applying Lemma 23.2 and the remark following it, with $W = \Omega \subseteq B$, we get a general Hartogs figure $\tilde{P}$ in $\mathbb{C}^2$ such that $\tilde{H} \subseteq \Omega$, $\tilde{P} \not\subseteq \Omega$, and $\tilde{P} \subseteq B$. Set $K = P \cap \Phi^{-1}(\tilde{P} \cap (\hat{Y} \setminus Y))$. Since $(\hat{Y} \setminus Y) \cap \tilde{H}$ is empty, it follows that $K \subseteq P \setminus H$ and so $z_2 \neq 0$ on K. Since $\tilde{P} \subseteq B$ and $\tilde{P} \not\subseteq \Omega$, K is nonempty. Moreover, since Φ is a biholomorphism on the *closed* polydisk P, we can enlarge the domain of Φ in the z_2-variable and so, by rescaling in z_2, we may assume that K meets the set $\{|z_2| < 1\}$ at some point (a, b), say. Since the local maximum modulus principle holds on $\hat{Y} \setminus Y$, it follows, since Φ is a biholomorphism, that it also holds on K. Namely, if g is holomorphic on a neighborhood of K, then, since $K \cap \partial P = K \cap \{|z_2| = 1\}$,

$$|g|_K \leq |g|_{K\cap\{|z_2|=1\}}.$$

Now, applying this to $g(z) \equiv 1/z_2$ yields $1/|b| \leq 1$. This is a contradiction, and Theorem 23.1 follows. □

EXERCISE 23.6. Let Y be the set

$$\{(0, 0, w) : |w| = 1\}$$

in $\mathbb{C}^3$. Thus Y is a circle lying on a complex line in $\mathbb{C}^3$.
(a) Show that $\hat{Y} = \{(0, 0, w) : |w| \leq 1\}$.
(b) Choose a small ball B in $\mathbb{C}^3$ centered at a point $(0, 0, w_0)$ in $\hat{Y} \setminus Y$. Show that $B \setminus \hat{Y}$ is not pseudoconvex.
(c) Deduce that the direct analogue of of Theorem 23.1 fails if $\mathbb{C}^2$ is replaced by $\mathbb{C}^3$.

4 Maximum Modulus Algebras and Pseudoconvexity

Let Y be a set in $\mathbb{C}^2$ lying over the unit circle Γ in the λ-plane. Put $X = \hat{Y} \setminus Y$, $D = \{|\lambda| < 1\}$ and let A be the restrictions to X of all polynomials in λ and w. Put $\pi =$ the map $(\lambda, w) \mapsto \lambda$ of $\mathbb{C}^2 \to \mathbb{C}$. It follows from the local maximum principle on $\hat{Y}$ that (A, X, D, π) is a maximum modulus algebra on X. We saw in Theorem 23.1 that, locally on X, $\mathbb{C}^2 \setminus X$ is pseudoconvex. We now consider a result that goes in the opposite direction. In order to formulate the result, we need to define a strong version of pseudoconvexity in the sense of Levi.

Definition 23.3. Let Ω be a bounded domain in $\mathbb{C}^2$, defined by $\{\rho < 0\}$, with ρ continuous. Fix $z^0 \in \partial\Omega$ such that ρ is C^2 on a neighborhood of z^0. We say that $\partial\Omega$ is *strictly pseudoconvex* at z^0 if we have

$$(8) \qquad \sum_{j,k=1}^{N} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right)_0 \zeta_j \bar{\zeta}_k > 0 \qquad \forall \text{ complex tangents } \zeta \neq 0 \in T_{z^0}.$$

Remark. This definition is analogous to strict convexity in $\mathbb{R}^N$. In fact, the next lemma, due to R. Narasimhan, shows that there is more than an analogy here.

Lemma 23.3. *Suppose that $\partial\Omega$ is strictly pseudoconvex at z^0, where $\Omega \subseteq \mathbb{C}^n$. Then there exists an open neighborhood V of z^0 in $\mathbb{C}^n$ and a biholomorphism ϕ of V onto an open set in $\mathbb{C}^n$ such that $\phi(\Omega \cap V)$ is strictly convex in $\mathbb{C}^n$.*

Remark. Suppose that z^0, Ω, and V are as in the lemma. Then there exists a function ψ holomorphic in V such that

$$Z(\psi) = \{z \in V : \psi(z) = 0\}$$

is a complex submanifold of V that passes through z^0 and satisfies

$$Z(\psi) \setminus \{z^0\} \subset \mathbb{C}^n \setminus \bar{\Omega}. \tag{9}$$

In fact, if Ω is strictly convex at z^0, then for ψ we can take the the affine complex linear function F such that $\{\mathrm{Re} F = 0\} = z^0 + T_{z^0}(\partial\Omega)$. In the general case, we apply the lemma and we set $\psi = F \circ \phi$, where F is the affine linear function for the strictly convex image $\phi(\Omega \cap V)$.

Proof of Lemma 23.3. Without loss of generality we may assume that $z^0 = 0$ and that $T_{z^0}(\partial\Omega) = \{x_n = 0\}$. We write

$$\rho(z) = x_n + \mathrm{Re}\Big(\sum_{1\le j,k\le n} \alpha_{jk} z_j z_k\Big) + \sum_{1\le j,k\le n} c_{jk} z_j \bar{z}_k + o(|z|^2),$$

where (c_{jk}) is Hermitian positive definite. We make a quadratic change of coordinates, putting

$$w_j = z_j,\ 1 \le j < n \qquad \text{and} \qquad w_n = z_n + \sum_{1\le j,k\le n} \alpha_{jk} z_j z_k.$$

Then $z \mapsto w \equiv \phi(z)$ gives a biholomorphism near $z = 0$, and in the w-coordinates

$$\rho = \mathrm{Re}\, w_n + \sum_{1\le j,k\le n} c_{jk} w_j \bar{w}_k + o(|w|^2).$$

Writing $w_j = u_j + iv_j$, $1 \le j \le n$, we see that the real Hessian H of $\rho \circ \phi^{-1}$ at $w = 0$, with respect to the real coordinates $u_1, v_1, u_2, \cdots, u_n, v_n$, is given by $H(u_1, v_1, u_2, \cdots, u_n, v_n) = \sum_{1\le j,k\le n} c_{jk} w_j \bar{w}_k$. Hence, since (c_{jk}) is Hermitian positive definite, it follows that H is (real) positive definite—this yields the desired strict convexity. $\square$

We shall denote the projection map of $\mathbb{C}^2$ to the first coordinate by π, i.e., $\pi(\lambda, w) = \lambda$. In the proof of the next theorem, we shall view π as a map restricted to a subset K of $\mathbb{C}^2$, and we shall write $\pi^{-1}(S) = \{(\lambda, w) \in K : \pi(\lambda) \in S\}$.

Theorem 23.4. *Let Ω be a domain in the product set $\{|\lambda| < 1\} \times \mathbb{C} \subset \mathbb{C}^2$ so that $\partial\Omega \cap \{|\lambda| < 1\}$ is smooth and strictly pseudoconvex at each point. Put $K = [\{|\lambda| < 1\} \times \mathbb{C}] \setminus \Omega$. Assume that K is bounded in $\mathbb{C}^2$. Denote by A the algebra of restrictions to K of all polynomials in λ and w. Then (A, K, D, π) is a maximum modulus algebra on K.*

Proof. We first fix a point (λ_0, w_0) in K that lies on $\partial\Omega \cap \{|\lambda| < 1\}$. Since $\partial\Omega$ is strictly pseudoconvex at (λ_0, w_0), we can use the remark to obtain a neighborhood V of (λ_0, w_0) in $\mathbb{C}^2$ as well as a function ψ holomorphic in V such that $Z(\psi)$ passes through (λ_0, w_0), and otherwise lies totally outside of $\bar{\Omega}$. We write $\Sigma = Z(\psi)$. Let us denote by Λ the restriction of the coordinate function λ to Σ.

Case 1. Λ is not a constant.

Then there exists $\epsilon > 0$ such that the image of Σ under Λ contains the disk $\{|\lambda - \lambda_0| \leq \epsilon\}$. We define the region

$$W = \{|\Lambda - \lambda_0| < \epsilon\}$$

on Σ. Then

$$\partial W = \{|\Lambda - \lambda_0| = \epsilon\}$$

on Σ. Since $\Sigma \setminus \{(\lambda_0, w_0)\}$ lies outside $\bar{\Omega}$, $\Sigma \subseteq K$. Hence $\partial W \subseteq K$, and so

$$\partial W \subseteq \pi^{-1}(\{|\lambda - \lambda_0| = \epsilon\}).$$

Now fix a polynomial Q in λ and w. The restriction of Q to Σ is analytic on Σ. The maximum principle on W then gives

$$|Q(\lambda_0, w_0)| \leq \max_{\partial W} |Q| \leq \max_{\pi^{-1}(\{|\lambda-\lambda_0|=\epsilon\})} |Q|.$$

Case 2. Λ is constant on Σ.

Then $\Lambda \equiv \lambda_0$ on Σ, so Σ is an open set on the complex line $\{\lambda = \lambda_0\}$. Since $\Sigma \setminus \{(\lambda_0, w_0)\}$ lies outside $\bar{\Omega}$, $\Sigma \setminus \{(\lambda_0, w_0)\}$ lies in the interior of K, and so we can choose a region W on Σ such that $(\lambda_0, w_0) \in W$ and ∂W is a compact subset of $\mathrm{int}(K)$. It follows that there exists $\epsilon > 0$ such that, for each $(\lambda, w_1) \in \partial W$, the entire horizontal disk

$$\{(\lambda_0 + \tau, w_1) : |\tau| \leq \epsilon\}$$

is contained in K. The boundary of the disk,

$$\{(\lambda_0 + \tau, w_1) : |\tau| = \epsilon\},$$

then is contained in

$$\pi^{-1}(\{|\lambda - \lambda_0| = \epsilon\}).$$

Fix a polynomial $Q(\lambda, w)$. Then

$$|Q(\lambda_0, w_0)| \leq |Q(\lambda_0, w_1)| \tag{10}$$

for some $(\lambda_0, w_1) \in \partial W$. Then, by the maximum principle on the horizontal complex line through (λ_0, w_1),

$$|Q(\lambda_0, w_1)| \leq |Q(\lambda_0 + \tau, w_1)| \tag{11}$$

for some τ with $|\tau| = \epsilon$. It follows from (10) and (11) that $|Q(\lambda_0, w_0)| \leq \max_{\pi^{-1}(\{|\lambda-\lambda_0|=\epsilon\})} |Q|$.

We have completed the case of a point (λ_0, w_0) in K that lies on $\partial\Omega \cap \{|\lambda| < 1\}$. The case of $(\lambda_0, w_0) \in K$, but $(\lambda_0, w_0) \notin \partial\Omega$, clearly reduces to the previous one.

Finally, it remains to show that π is a proper map of K onto $D = \{|\lambda| < 1\}$. That π is proper is clear since K is a bounded and relatively closed subset of $D \times \mathbb{C}$. In particular, $\pi(K)$ is a closed subset of D. To see that π maps onto D it therefore suffices to show that $\pi(K)$ is an open subset of D. We have $K = \operatorname{int}(K) \cup (\partial\Omega) \cap (D \times \mathbb{C}))$. $\pi(\operatorname{int}(K))$ is clearly open and so it suffices to show that $\pi(\partial\Omega \cap (D \times \mathbb{C}))$ is contained in the interior of $\pi(K)$. This is clear from the discussions of Case 1 and of Case 2 above. This completes the proof. □

5 Levi-Flat Hypersurfaces

Consider a region Ω in $\mathbb{R}^N$ defined by an inequality $\rho(x) < 0$, where ρ is as in Section 1. Fix $x^0 \in \partial\Omega$. If $\partial\Omega$ is flat, i.e., if a neighborhood of x^0 on $\partial\Omega$ lies on a hyperplane, then both Ω and its complementary region are convex near x^0. The complementary region is given by the inequality $\rho(x) > 0$ near x^0, so we have, in view of (2), the two relations

$$\sum_{j,k=1}^{N} \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}\right)_0 \xi_j\xi_k \geq 0 \qquad \forall \xi \in T_{x^0}, \qquad \text{and} \tag{13}$$

$$\sum_{j,k=1}^{N} \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}\right)_0 \xi_j\xi_k \leq 0 \qquad \forall \xi \in T_{x^0}. \tag{14}$$

Hence we have

$$\sum_{j,k=1}^{N} \left(\frac{\partial^2\rho}{\partial x_j \partial x_k}\right)_0 \xi_j\xi_k = 0 \qquad \forall \xi \in T_{x^0}. \tag{15}$$

Now let Ω be a domain in $\mathbb{C}^n$ defined by $\rho(z) < 0$, and fix $z^0 \in \partial\Omega$, where $\partial\Omega$ is smooth near z^0. By analogy with (15), we make the following definition.

Definition 23.4. $\partial\Omega$ is *Levi-flat at* z^0 if we have

$$\sum_{j,k=1}^{n} \left(\frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\right)_0 \zeta_j\bar{\zeta}_k = 0 \tag{15$'$}$$

for every complex tangent vector ζ to $\partial\Omega$ at z^0.

EXERCISE 23.7. Let ϕ be a function analytic on $\mathbb{C}^2$ and let Ω be the domain

$$|\phi(z_1, z_2)| < 1$$

in $\mathbb{C}^2$. Fix $z^0 \in \partial\Omega$ such that $\partial\Omega$ is smooth near z^0. Show that then $\partial\Omega$ is Levi-flat at z^0.

For simplicity, we restrict ourselves to the case where $n = 2$ for the rest of this section. Now let Σ be an analytic disk in $\mathbb{C}^2$ given by

$$z = f(\lambda), \qquad |\lambda| < 1, \tag{16}$$

where $f(\lambda) = (f_1(\lambda), f_2(\lambda))$ with each f_j analytic in $\{|\lambda| < 1\}$. We put

$$\frac{df}{d\lambda} = (\frac{df_1}{d\lambda}, \frac{df_2}{d\lambda}).$$

Theorem 23.5. *Let Ω be a region in $\mathbb{C}^2$ and fix $z^0 \in \partial\Omega$ with $\partial\Omega$ smooth near z^0. Assume that there exists an analytic disk, $z = f(\lambda)$, with $f(0) = z^0$, which is contained in $\partial\Omega$, and $\frac{df}{d\lambda}(0) \neq 0$. Then $\partial\Omega$ is Levi-flat at z^0.*

PROOF. Let Ω be given by $\{\rho(z) < 0\}$ with ρ smooth in a neighborhood of z^0. Then

$$\rho(f(\lambda)) = 0, \ |\lambda| < 1.$$

Differentiating, we get, writing $f_j' = df_j/d\lambda$, $\partial/\partial\lambda(\rho(f(\lambda)) = 0$ or

$$\sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(f(\lambda)) f_j'(\lambda) = 0.$$

It follows that $f'(0)$ is a complex tangent to $\partial\Omega$ at z^0. Differentiating the last equation with respect to $\bar{\lambda}$, we get

$$\sum_j [\sum_k \frac{\partial^2}{\partial \bar{z}_k \partial z_j}(f(\lambda)) f_j'(\lambda)\overline{f_k'(\lambda)}] = 0, \qquad \text{or}$$

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(f(\lambda)) f_j'(\lambda)\overline{f_k'(\lambda)} = 0, \tag{17}$$

In view of (1′), the totality of complex tangent vectors to $\partial\Omega$ at z^0 is a one-dimensional complex subspace of $\mathbb{C}^2$. Fix such a complex tangent vextor ζ. Then there exists a $\lambda \in \mathbb{C}$ such that $\zeta = \lambda f'(0)$. Hence we get

$$\sum_{j,k} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(0)\zeta_j\overline{\zeta_k} = \sum_{j,k} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\right)_0 |\lambda|^2 f_j'(0)\overline{f_k'(0)} = 0,$$

in view of (17). Thus $\partial\Omega$ is Levi-flat at z^0, as claimed. □

Levi-flat hypersurfaces arise narturally in the study of polynomial hulls.

EXAMPLE 23.2. Let $\mathbb{T}$ denote the torus $\{|z_1| = 1, |z_2| = 1\}$ in $\mathbb{C}^2$. Put $\Omega = \text{int}\Delta^2$. Then $\hat{\mathbb{T}} = \bar{\Omega}$. Theorem 23.5 (or direct calculation) yields that $\partial\Omega$ is Levi-flat at each point (λ_0, w_0) in $\partial\Omega$ with $|\lambda_0| < 1$.

This example points the way to the following theorem.

Theorem 23.6. *Let Y be a compact set in $\mathbb{C}^2$ lying over the circle $\{|z_1| = 1\}$. Let Ω be the interior of $\hat{Y}$ (Ω is contained in $\{|z_1| < 1\}$). Then $\partial\Omega$ is Levi-flat at each point $z^0 = (z_1^0, z_2^0)$ with $|z_1^0| < 1$ at which $\partial\Omega$ is smooth near z^0.*

PROOF. Fix $z^0 \in \partial\Omega$ with $|z_1^0| < 1$. Choose a defining function ρ for Ω such that

$$\Omega = \{\rho(z) < 0\},$$

ρ is smooth in a neighborhood of z^0 in $\mathbb{C}^2$, and $(\nabla\rho)_0 \neq 0$.

Let B be a small ball centered at z^0. We put $\omega = B \setminus \hat{Y}$. By Theorem 23.1, ω is pseudoconvex. We choose a defining function $\tilde{\rho}$ for ω such that $\tilde{\rho}$ coincides with $-\rho$ in a neighborhood of z^0 in $\mathbb{C}^2$. Since ω is pseudoconvex, we have

$$\sum_{j,k} \left(\frac{\partial^2 \tilde{\rho}}{\partial z_j \partial \bar{z}_k} \right)_0 \zeta_j \bar{\zeta}_k \geq 0$$

for every complex tangent ζ to $\partial\Omega$ at z^0 and hence

$$\sum_{j,k} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right)_0 \zeta_j \bar{\zeta}_k \leq 0 \tag{18}$$

for every such ζ.

On the other hand, since Ω is the interior of the polynomially convex set $\hat{Y}$, Ω is a Runge domain and hence pseudoconvex, and therefore Ω is pseudoconvex in the sense of Levi at each of its smooth points z^0. (See the Appendix for these implications.) It follows that

$$\sum_{j,k} \left(\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k} \right)_0 \zeta_j \bar{\zeta}_k \geq 0 \tag{19}$$

for every complex tangent ζ to $\partial\Omega$ at z^0. Now (18) and (19) yield (15′); i.e., $\partial\Omega$ is Levi-flat at z^0. $\square$

24

Examples

Let Y be a compact set in $\mathbb{C}^n$ and fix a point $\zeta \in \mathbb{C}^n \setminus Y$. What is the "reason" that ζ lies in the hull of Y, or, in other words, how can we account for the inequality

$$|P(\zeta)| \leq \max_Y |P| \tag{1}$$

for all polynomials P on $\mathbb{C}^n$?

The simplest explanation for (1) would be to reduce the inequality to the maximum principle for analytic functions on some analytic variety. Suppose that there exists an analytic variety Σ such that:

(i) $\zeta \in \Sigma$;
(ii) the boundary of Σ, $\partial\Sigma$, is contained in Y; and
(iii) $\Sigma \cup \partial\Sigma$ is compact.

If, then, P is any polynomial, the restriction of P to Σ is analytic, and so (1) is a consequence of the maximum principle for analytic functions on Σ.

In the 1950s, the question was asked whether, indeed, for each compact set Y in $\mathbb{C}^n$ with $\hat{Y} \neq Y$, some analytic variety of positive dimension is contained in $\hat{Y}$. A counterexample was given by Stolzenberg in [St2] in 1963.

In this chapter we shall give a number of examples related to the problem of the existence of analytic varieties in hulls.

Definition 24.1. Let X be a compact subset of $\mathbb{C}^n$. Then $R_0(X)$ denotes the algebra of continuous functions f on X of the form $f = A/B$, where A and B are polynomials and B has no zero on X. $R(X)$ denotes the closure of $R_0(X)$ in the uniform norm over X.

Definition 24.2. Let X be a compact subset of $\mathbb{C}^n$. The *rationally convex hull of* X, denoted $h_r(X)$, is the set $\{z \in \mathbb{C}^n :$ for all polynomials p in $\mathbb{C}^n$, if $p(z) = 0,$ then p has a zero on $X\}$.

The reader can verify that (a) $h_r(X) \subseteq \hat{X}$ and (b) the maximal ideal space of $R(X)$ can be naturally identified with $h_r(X)$.

EXAMPLE 24.1. Let S be a closed subset of $\{|\zeta| \leq 1\}$ that contains the unit circle. Denote by $D_1, D_2, \ldots$ the components of the complement of S in $\{|\zeta| < 1\}$. For each i, put

$$H_i = \{(z, w) \in \mathbb{C}^2 : z \in D_i, |w| = 1\}$$

and

$$K_i = \{(z, w) \in \mathbb{C}^2 : |z| = 1, w \in D_i\}.$$

Δ^2 denotes the closed bidisk and $\partial\Delta^2$ its topological boundary. We can picture each H_i or K_i as a solid torus in $\partial\Delta^2$. We denote

$$X_S = \partial\Delta^2 \setminus \bigcup_{i=1}^{\infty} H_i \cup K_i.$$

Thus X_S is obtained from $\partial\Delta^2$ by removing a family of solid tori. Another representation of X_S is

$$X_S = \{(z, w) : z \in S, |w| = 1\} \cup \{(z, w) : |z| = 1, w \in S\}.$$

Claim 1. Assume that S has no interior. Then $h_r(X)$ contains no analytic disk.

PROOF. Suppose that E is an analytic disk contained in $h_r(X_S)$. Either z or w is not a constant on E. Suppose that z is not constant. Then $z(E)$ contains interior in $\mathbb{C}$ and so the z-projection of $h_r(X_S)$ has interior points. On the other hand, this z-projection is contained in S. To see this, consider $(z_0, w_0) \in h_r(X_S)$. If $z_0 \notin S$, then $z - z_0 \neq 0$ on S. But $z - z_0$ vanishes at (z_0, w_0), and this contradicts the definition of $h_r(X_S)$. So $z_0 \in S$, as claimed. Since by hypothesis S has no interior, we have a contradiction. Thus E cannot exist. □

Of course, given S it may be that $h_r(X_S)$ coincides with X_S, and in this case Claim 1 gives no information. We now shall construct S such that S has no interior and $h_r(X_S) \neq X_S$.

Let $S, D_i, H_i, K_i, i = 1, 2, \ldots$ be as above. For each n, we put

$$Y_n = \bigcup_{i=1}^{n} H_i \cup K_i.$$

Claim 2. Fix $p \in \Delta^2$. If $p \notin h_r(X_S)$, then for some n, $p \in \hat{Y}_n$ in the sense that for every polynomial f, $|f(p)| \leq \sup_{Y_n} |f|$.

PROOF. Since $p \notin h_r(X_S)$, there exists a polynomial Q with $Q(p) = 0$ and $Q \neq 0$ on X_S. Since Q is continuous and $X_S = \partial\Delta^2 \setminus \bigcup_{i=1}^{\infty} H_i \cup K_i$, we can choose n with $Q \neq 0$ on $\partial\Delta^2 \setminus \bigcup_{i=1}^{n} H_i \cup K_i$.

Denote by V the connected component of the zero-set of Q containing p. Then $V \cap \partial\Delta \subseteq \bigcup_{i=1}^{n} H_i \cup K_i$. If now f is a polynomial, the maximum principle applied to V gives

$$|f(p)| \leq \max_{V \cap \partial\Delta} |f| \leq \sup |f|,$$

where the sup is taken over $\bigcup_{i=1}^{n} H_i \cup K_i$, as claimed. $\square$

Claim 3. There exists a sequence $D_1, D_2, \ldots$ of disjoint open subsets of $\{|\zeta| < 1\}$ such that, if we put

$$S = \{|\zeta| < 1\} \setminus \bigcup_{i=1}^{\infty} D_i,$$

and define H_i, K_i as earlier, then we have:

$$S \text{ lacks interior, and} \tag{2}$$

$$(0, 0) \in h_r(X_S). \tag{3}$$

Proof. We choose a countable dense subset $\{a_j\}$ of $\{|\zeta| < 1\}$ avoiding 0.

We shall show that there exists a sequence $\{D_j\}$ of disjoint open disks contained in $\{|\zeta| < 1\}$ such that for each n we have

$$a_j \in \bigcup_{i=1}^{n} D_i \text{ for } 1 \leq j \leq n, \text{ and} \tag{4}$$

$$0 \notin \hat{Y}_n, \text{ where } Y_n = \bigcup_{i=1}^{n} H_i \cup K_i. \tag{5}$$

Fix r_1 with $2r_1/|a_1|^2 < 1$. Put $G(z, w) = (z - a_1)(w - a_1)/a_1^2$. Put $D_1 = \{|\zeta - a_1| < r_1\}$. Then $G(0, 0) = 1$ and, for $(z, w) \in H_1 \cup K_1$, $|G(z, w)| \leq 2r_1/|a_1|^2 < 1$. So (4) and (5) hold for $n = 1$.

Suppose now that disjoint open disks $D_1, D_2, \ldots, D_s$ have been chosen so that (4) and (5) hold for $n = 1, 2, \ldots, s$, and also that ∂D_i does not meet $\{a_j\}$ for $i = 1, 2, \ldots, s$. Let a be the first a_j not contained in $\cup_{i=1}^{s} D_i$. It follows from (5) for $n = s$ that there exists a polynomial P with $|P(0, 0)| > 1$ and $|P| \leq 1/2$ on $\cup_{i=1}^{s} H_i \cup K_i$. (Why?) Put $\lambda = \max_{\partial\Delta^2} |P|$. Choose k such that

$$\frac{(\frac{1}{2})^k 4}{|a|^2} < 1$$

and choose r with

$$\frac{\lambda^k 2r}{|a|^2} < 1.$$

Put $D_{s+1} = \{|\zeta - a| < r\}$. We may assume that D_{s+1} fails to meet $\cup_{i=1}^{s} D_i$ and that ∂D_{s+1} does not meet $\{a_j\}$. Put

$$Q(z, w) = (P(z, w))^k (z - a)(w - a)/a^2.$$

Then $|Q(0)| > 1$ and on $\cup_{i=1}^{s} H_i \cup K_i$,

$$|Q|^2 \leq \frac{(\frac{1}{2})^k 4}{|a|^2} < 1.$$

So $0 \notin \hat{Y}_n$ for $n = 1, 2, \ldots, s$.

On $H_{s+1} \cup K_{s+1}$, either $|z - a| < r$ or $|w - a| < r$, so $|Q| < \frac{\lambda^k 2r}{|a|^2} < 1$. Thus $0 \notin \hat{Y}_{s+1}$, and so (5) holds for $n = s + 1$. By choice of D_{s+1}, $a_{s+1} \in \cup_{i=1}^{s+1} D_i$. So (4) and (5) hold for $D_1, \ldots, D_s, D_{s+1}$. Thus by induction, the desired sequence $\{D_i\}$ exists satisfying (4) and (5) for each n. We now put $S = \{|\zeta| \leq 1\} \setminus \cup_{i=1}^{\infty} D_i$.

Because of Claim 2, together with (5), we have that $0 \in h_r(X_S)$. Also, $\cup_{i=1}^{\infty} D_i$ contains each a_j, and hence S lacks interior. This gives Claim 3.

Finally, Claim 1 gives that $h_r(X_S)$ contains no analytic disk. We have proved

Theorem 24.1. *There exists S such that $h_r(X_S) \neq X_S$ and $h_r(X_S)$ contains no analytic disk.*

It is shown in the Appendix that, if X is a compact set in $\mathbb{C}^n$, then $R(X)$ is generated by $n + 1$ functions.

Exercise 24.1. Let S be as in Theorem 24.1. Let g_1, g_2, g_3 be three functions in $R(X_S)$, generating that algebra. Denote by Y the image in $\mathbb{C}^3$ of X_S under the map $g = (g_1, g_2, g_3)$. Show that $\hat{Y} \neq Y$ and $\hat{Y}$ contains no analytic disk.

Example 24.2. Let $\mathfrak{A}$ be a uniform algebra on a compact space X, and let $\mathcal{M}$ be the maximal ideal space of $\mathfrak{A}$. Fix $f \in \mathfrak{A}$. Assume that $f(X)$ is the unit circle and that $f(\mathcal{M})$ is the unit disk.

In Chapter 20 we studied this situation in a special case, and in Theorem 20.2 we found a class of such algebras where $\mathcal{M}$ must contain analytic disks.

However, we have the following result of Cole [Co1]:

Theorem 24.2. *There exists a uniform algebra $\mathfrak{A}$ on a compact metric space X, with maximal ideal space $\mathcal{M}$ and $f \in \mathfrak{A}$ such that*

(6) *$f(X)$ is the unit circle.*

(7) *$f(\mathcal{M})$ is $\{|z| \leq 1\}$.*

(8) *$\mathcal{M}$ contains no analytic disk.*

Proof. Let Δ_0 be the closed unit disk and let $\Delta_1, \Delta_2, \ldots$ be a sequence of copies of the disk $\{|z| \leq 2\}$. Let $\Pi = \Delta_0 \times \Delta_1 \times \Delta_2 \times \cdots$ be the topological product of all of these disks. Then Π is a compact metrizable space. A point of Π will be denoted by $(z, \zeta_1, \zeta_2, \ldots)$, where $|z| \leq 1$ and $|\zeta_j| \leq 2$ for all j. We denote the coordinate functions by $z, \zeta_1, \zeta_2, \ldots$. Each coordinate function is continuous on Π.

Let $\{a_i\}$ be a countable dense subset of $\{|z| < 1\}$. We denote by Y the subset of Π consisting of all points $(z, \zeta_1, \zeta_2, \ldots)$ satisfying

$$\zeta_1^2 = z - a_1, \zeta_2^2 = z - a_2, \cdots.$$

Y is thus the common null-set of a family of continuous functions on Π, and so Y is closed and hence compact. Let $\mathfrak{A}(Y)$ denote the uniform algebra on Y spanned by all of the polynomials $P(z, \zeta_1, \zeta_2, \ldots, \zeta_n)$, $n = 1, 2, \ldots$ in the coordinate functions.

Put $X = \{(z, \zeta_1, \zeta_2, \ldots) \in Y : |z| = 1\}$. We claim that X is a boundary for $\mathfrak{A}(Y)$, in the sense of Definition 9.1.

Fix n and consider the variety $V_n = \{(z, \zeta_1, \zeta_2, \ldots, \zeta_n) : \zeta_j^2 = z - a_j, 1 \leq j \leq n\} \subseteq \mathbb{C}^{n+1}$. Let K denote the polydisk in $\mathbb{C}^{n+1}$ consisting of all points $(z_0, z_1, \ldots, z_n)$ with $|z_0| \leq 1$ and $|z_j| \leq 2$ for $j = 1, 2, \ldots, n$. Then $V_n \cap K^o$ is an analytic subvariety of K^o, where K^o denotes the interior of K. Let $(z^0, \zeta_1^0, \zeta_2^0, \ldots, \zeta_n^0)$ be a boundary point of $V_n \cap K^o$. If $|z^0| \neq 1$, then $|\zeta_j^0| = 2$ for some j. But $|\zeta_j^0|^2 = |z^0 - a_j| < 2$, so this cannot occur. Hence

$$\partial(V_n \cap K^o) \subseteq \{(z, \zeta_1, \zeta_2, \ldots, \zeta_n) : |z| = 1\}. \tag{9}$$

Consider a polynomial P on $\mathbb{C}^{n+1}$ and let $g = P(z, \zeta_1, \zeta_2, \ldots, \zeta_n)$ be the corresponding element of $\mathfrak{A}(Y)$. Fix $y = (z^0, \zeta_1^0, \zeta_2^0, \ldots) \in Y$. Then $(z^0, \zeta_1^0, \zeta_2^0, \ldots, \zeta_n^0) \in V_n \cap K$. By the maximum principle on V_n and (9), there exists $(z', \zeta_1', \zeta_2', \ldots, \zeta_n') \in V_n$ with $|z'| = 1$ such that

$$|g(y)| = |P(z^0, \zeta_1^0, \zeta_2^0, \ldots, \zeta_n^0)| \leq |P(z', \zeta_1', \zeta_2', \ldots, \zeta_n')|.$$

Next, we can (clearly) choose $\zeta_{n+1}', \zeta_{n+2}', \ldots$ so that $y' = (z', \zeta_1', \zeta_2', \ldots, \zeta_n', \zeta_{n+1}', \zeta_{n+2}', \ldots) \in Y$. Then $|g(y)| \leq |g(y')|$. Since $y' \in X$, it follows that X is a boundary for $\mathfrak{A}(Y)$, as claimed.

The restriction of $\mathfrak{A}(Y)$ to X is then a uniform algebra $\mathfrak{A}$ on X. It is easy to see that the maximal ideal space of $\mathfrak{A}$ is Y, and we leave it to the reader to verify this.

We take f to be the coordinate function z and let $\mathcal{M}$ denote the maximal ideal space of $\mathfrak{A}$. Then $f(X)$ is the unit circle and $f(\mathcal{M})$ is the unit disk.

We assert that $\mathcal{M}$ contains no analytic disk. Suppose that there were such a disk E. This means that there is a continuous one-one map Φ of $\{|\lambda| < 1\}$ onto E such that $h \circ \Phi$ is analytic on $\{|\lambda| < 1\}$ for every $h \in \mathfrak{A}$. We first suppose that f is not constant on E and put $F = f \circ \Phi$. Then F is a nonconstant analytic function and so $F(\{|\lambda| < 1/2\})$ contains an open disk. In that disk there are infinitely many of the a_j. For each such j choose λ_j in $\{|\lambda| < 1\}$ with $F(\lambda_j) = a_j$.

Fix j. Since the coordinate function ζ_j satisfies $\zeta_j^2 = z - a_j$, we have $\zeta_j^2 = f - a_j$, and so $(\zeta_j \circ \Phi)^2 = F - a_j$. Hence the derivative F' of F vanishes at λ_j. Hence F' vanishes infinitely often in $\{|\lambda| < 1/2\}$ and so F is a constant. This is a contradiction. So f is constant on E.

Thus for some $a \in \Delta_0$, $f^{-1}(a)$ contains E. We claim, however, that for each $z_0 \in \Delta_0$, $f^{-1}(z_0)$ is totally disconnected, and this will yield a contradiction. Let G_j be the two element group $\{-1, 1\}$ for $j = 1, 2, \cdots$ and let G be the topological product, $G = \prod_{j=1}^{\infty} G_j$. An element of G is a sequence $g = (g_1, g_2, \ldots)$, where each $g_j = 1$ or $= -1$. G is a compact, totally disconnected, Hausdorff space. We shall construct a homeomorphism of G onto $f^{-1}(z_0)$. Each point of $f^{-1}(z_0)$ has the form $x = (z_0, w_1, w_2, \ldots)$, where $w_j^2 = z_0 - a_j$ for all j. We only consider the case when $z_0 \neq a_j$ for all j, and so $w_j \neq 0$ for all j. The case $z_0 = a_j$ for some j is similar. Fix $x' = (z_0, w_1', w_2', \ldots) \in f^{-1}(z_0)$. For each $g = (g_1, g_2, \cdots) \in G$, map $g \to gx' = (z_0, g_1 w_1', g_2 w_2', \ldots)$. Then gx' again belongs to $f^{-1}(z_0)$. It is easy to verify that the map $g \to gx'$ maps G onto $f^{-1}(z_0)$, and that the map is one-one and continuous. Since G is compact, the map is a homeomorphism. Hence $f^{-1}(z_0)$ is totally disconnected, as claimed. We are done. □

Note. In an intuitive sense, the space Y in the preceding example is a Riemann surface lying over the unit disk, which "has a dense set of branch points." Taking this point of view, in the following example, we shall construct a variant of the space Y that lies in $\mathbb{C}^2$.

EXAMPLE 24.3. We next give an example of a hull without analytic structure. Let π denote the projection of $\mathbb{C}^2$ to $\mathbb{C}$ given by

$$\pi(z_1, z_2) = z_1.$$

Theorem 24.3. *There exists a compact subset Y of $\mathbb{C}^2$ with $\pi(Y) = \{|z| = 1\}$ and $\pi(\hat{Y}) = \{|z| \leq 1\}$ such that $\hat{Y} \setminus Y$ contains no analytic disk.*

Note. By a change of variable we may replace the unit circle by the circle $\{|z| = 1/2\}$ and the unit disk by $\{|z| \leq 1/2\}$; we shall prove Theorem 24.3 for this case. The convenience that results is that for $|a|, |b| \leq 1/2$ we have $|a - b| \leq 1$.

PROOF OF THEOREM 24.3. We denote by $a_1, a_2, \ldots$ the points in the disk $\{|z| \leq 1/2\}$, both of whose coordinates are rational. For $j = 1, 2 \ldots$, we denote by B_j the algebraic function

$$B_j(z) = (z - a_1)(z - a_2) \cdots (z - a_{j-1})\sqrt{z - a_j}.$$

Fix constants $c_1, c_2, \ldots, c_n$ in $\mathbb{C}$ and put

$$g_n(z) = \sum_{j=1}^{n} c_j B_j(z).$$

We denote by $\Sigma(c_1, c_2, \ldots, c_n)$ the portion of the Riemann surface of g_n that lies in $\{|z| \leq 1/2\}$. In other words,

$$\Sigma(c_1, c_2, \ldots, c_n) = \{(z, w) : |z| \leq 1/2,\ w = w_j,\ j = 1, 2, \ldots, 2^n\},$$

where w_j, $j = 1, 2, \ldots, 2^n$ are the values of g_n at z. In the following discussion, we shall always assume, whenever z occurs, that

$$|z| \leq \tfrac{1}{2}. \tag{10}$$

□

Lemma 24.4. *There exist two sequences of positive constants c_j, $j = 1, 2, \ldots$ and ϵ_j, $j = 1, 2, \ldots$ such that $c_1 = 1/10$, $c_{j+1} \leq (1/10)c_j$, $j = 1, 2, \ldots$, and there exists a sequence of polynomials P_n in z and w, $n = 1, 2, \ldots$ such that:*

(11) $\{P_n = 0\} = \Sigma(c_1, c_2, \ldots, c_n), \qquad n = 1, 2, \ldots,$

(12) $\{|P_n| \leq \epsilon_n\} \subseteq \{|P_{n-1}| \leq \epsilon_{n-1}\}, \qquad n = 2, 3, \ldots;$ *and*

(13) *if* $|a| \leq 1/2$ *and* $|P_n(a, w)| \leq \epsilon_n$, *then there exists* w_n *with* $P_n(a, w_n) = 0$ *and* $|w - w_n| < 1/n$, $n = 1, 2, \ldots$.

PROOF. For $j = 1$, we take

$$c_1 = \frac{1}{10}, \qquad \epsilon_1 = \frac{1}{4}, \qquad P_1(z, w) = w^2 - \frac{1}{100}(z - a_1).$$

Then (11) and (13) hold for $n = 1$, and (12) is vacuous.

Suppose now that c_j, ϵ_j, P_j have been chosen for $j = 1, 2, \ldots, n$, so that our three conditions are satisfied for each j. We shall choose $c_{n+1}, \epsilon_{n+1}, P_{n+1}$.

We denote by $w_j(z)$, $j = 1, 2, \ldots, 2^n$, the roots of $P_n(z, \cdot) = 0$. To each constant $c \geq 0$ we assign the polynomial $P_c(z, w)$ defined so that the roots of $P_c(z, \cdot) = 0$ are $w_j(z) \pm cB_{n+1}(z)$, $j = 1, 2, \ldots, 2^n$, and $P_c(z, w)$ is monic in w. Thus

$$P_c(z, w) = \prod_{j=1}^{2^n} (w - [w_j(z) + cB_{n+1}(z)])(w - [w_j(z) - cB_{n+1}(z)])$$

$$= \prod_{j=1}^{2^n} [(w - w_j(z))^2 - c^2(B_{n+1}(z))^2].$$

By construction, the zero-set of $P_c(z, w)$ is $\Sigma(c_1, \ldots, c_n, c)$.

Choose $M > 0$ such that, if we put $\Delta_M = \{(z, w) : |z| \leq 1/2, |w| \leq M\}$, then

$$\{|P_c| < \frac{\epsilon_n^2}{2}\} \subseteq \Delta_M$$

for all c, $0 \leq c \leq 1$.

Claim. For sufficiently small positive c, we have

$$\{|P_c| \leq \frac{\epsilon_n^2}{2}\} \subseteq \{|P_n| < \epsilon_n\}. \tag{14}$$

PROOF CLAIM. Suppose that the claim is false. Then for arbitrarily small $c > 0$ there exists $\zeta_c \in \Delta_M$ with $|P_c(\zeta_c)| \leq \epsilon_n^2/2$ and $|P_n(\zeta_c)| \geq \epsilon_n$. Since Δ_M is compact, the set $\{\zeta_c\}$ has an accumulation point $\zeta^* \in \Delta_M$. As $c \to 0$, $P_c \to (P_n)^2$ uniformly on compact sets in $\mathbb{C}^2$, and so $|P_n^2(\zeta^*)| \leq \epsilon_n^2/2$ and $|P_n(\zeta^*)| \geq \epsilon_n$. This is false. Thus the claim follows.

Now fix c such that (14) is true, and such that $c < (1/10)c_n$. We have

$$P_c(z, w) = \prod_{j=1}^{2^{n+1}} (w - w'_j(z)),$$

where $w'_j(z)$ are the zeros of $P_c(z, \cdot)$. Hence, if $\epsilon > 0$ and if $|P_c(z, w)| < \epsilon$, then $|w - w'_j(z)| < \epsilon^{1/2^{n+1}}$ for some j.

It follows that we can choose ϵ_{n+1} with $\epsilon_{n+1} < \epsilon_n^2/2$ and such that $|P_c(z, w)| < \epsilon_{n+1}$ implies that there exists w_{n+1} with $P_c(z, w_{n+1}) = 0$ and $|w - w_{n+1}| < 1/(n+1)$. Putting $c_{n+1} = c$, and then, putting $P_{n+1} = P_c$ and choosing ϵ_{n+1} as above, we have that (11), (13), and (12) are satisfied for $j = 1, 2, \ldots, n+1$. The lemma now follows by induction. $\square$

Definition 24.3. With P_n, ϵ_n chosen as in Lemma 24.4, we put (recall that $\{|z| \leq 1/2\}$ is understood!)

$$X = \bigcap_{n=1}^{\infty} \{|P_n(z, w)| \leq \epsilon_n\}.$$

It follows from this definition that X is a compact polynomially convex subset of $\{|z| \leq 1/2\} \subseteq \mathbb{C}^2$. For each n we put

$$\Sigma_n = \{P_n = 0\} = \Sigma(c_1, \ldots, c_n),$$

where $c_1, c_2, \ldots$ is the sequence obtained in Lemma 24.4.

Lemma 24.5. *A point (z, w) belongs to X if and only if there exists a sequence $\{(z, w_n)\}$ with $(z, w_n) \in \Sigma_n$ for each n and $w_n \to w$ as $n \to \infty$.*

Proof. Fix (z, w) and assume that such a sequence $\{(z, w_n)\}$ exists. Fix n_0. Because of (12),

$$\{|P_k| \leq \epsilon_k\} \subseteq \{|P_{n_0}| \leq \epsilon_{n_0}\}$$

if $k > n_0$. Since $P_k(z, w_k) = 0$ for each k,

$$(z, w_k) \in \{|P_{n_0}| \leq \epsilon_{n_0}\}$$

for each $k > n_0$. Hence $(z, w) \in \{|P_{n_0}| \leq \epsilon_{n_0}\}$. Since this holds for each n_0, $(z, w) \in X$.

Conversely, assume that $(z, w) \in X$. Fix n. Then $\{|P_n(z, w)| \leq \epsilon_n\}$. By (13) there hence exists w_n with $(z, w_n) \in \Sigma_n$ and $|w - w_n| < 1/n$. Thus $\{(z, w_n)\}$ is a sequence as required. $\square$

Lemma 24.6. *Let Ω be a region contained in $\{|z| < 1/2\}$. There does not exist a continuous function f on Ω whose graph $\{(z, f(z)) : z \in \Omega\}$ is contained in X.*

Proof. Suppose that such a function f exists. We choose a rectangle: $s_1 \leq \operatorname{Re} z \leq s_2$, $t_1 \leq \operatorname{Im} z \leq t_2$ contained in Ω, with s_1, s_2, t_1, t_2 irrational numbers. Let γ

denote the boundary of this rectangle. Denote by z_1 the midpoint of the left-hand edge of γ, by z_0 the midpoint of the right-hand edge, and let γ_1 denote the punctured curve $\gamma \setminus \{z_1\}$.

For each j, $B_j(z) = (z - a_1)\cdots(z - a_{j-1})\sqrt{z - a_j}$ has two single-valued continuous branches defined on γ_1. If a_j lies outside γ, then each branch extends continuously to γ, while if a_j lies inside γ, each branch has a jump discontinuity at z_1. By construction, no a_j lies on γ. We choose one of these two branches, arbitrarily, and call it β_j. Then $|\beta_j|$ is single-valued and nonvanishing on γ.

Let n be the smallest index such that a_n lies inside γ, and let $c_1, c_2, \ldots, c_n$ be the constants constructed in Lemma 24.4. The algebraic function $\sum_{j=1}^n c_j B_j$ has on γ_1 the 2^n branches

$$\sum_{j=1}^{n} c_j \rho_j \beta_j,$$

where each constant $\rho_j = 1$ or $= -1$.

Definition 24.4. $\mathcal{K}$ is the collection of all 2^n functions $\sum_{j=1}^n c_j\rho_j\beta_j$ on γ_1, where each ρ_j is a constant $= 1$ or $= -1$.

Claim 1. Fix $z \in \gamma_1$. There exists $k \in \mathcal{K}$, depending on z, such that

$$|f(z) - k(z)| \leq \tfrac{1}{4}|\beta_n(z)|c_n. \tag{15}$$

Proof of Claim. Since $(z, f(z)) \in X$, Lemma 24.5 gives w_N such that $(z, w_N) \in \Sigma_N$ and $R(z) = f(z) - w_N$ satisfies

$$|R(z)| \leq \tfrac{1}{10}|\beta_n(z)|c_n. \tag{16}$$

Thus $f(z) = \sum_{j=1}^N c_j\rho_j(z)\beta_j(z) + R(z)$, where each $\rho_j(z) = 1$ or $= -1$. So

$$f(z) = \sum_{j=1}^{n} c_j\rho_j(z)\beta_j(z) + \sum_{j=n+1}^{N} c_j\rho_j(z)\beta_j(z) + R(z)$$

$$= k(z) + \sum_{j=n+1}^{N} c_j\rho_j(z)\beta_j(z) + R(z),$$

where $k \in \mathcal{K}$. Then

$$|f(z) - k(z)| \leq \sum_{j=n+1}^{N} c_j|\beta_j(z)| + |R(z)|. \tag{17}$$

For each j,

$$|\beta_{j+1}(z)| = |(z - a_1)\cdots(z - a_j)||\sqrt{z - a_{j+1}}|$$

$$\leq |(z - a_1)\cdots(z - a_j)|$$

$$\leq |(z-a_1)\cdots(z-a_{j-1})|\sqrt{|z-a_j|} = |\beta_j(z)|,$$

where we have used that we are working in the disk $\{|z| \leq 1/2\}$. So

$$\sum_{j=n+1}^{N} c_j|\beta_j(z)| \leq \sum_{j=n+1}^{N} c_j|\beta_n(z)|$$

$$\leq |\beta_n(z)|[\frac{c_n}{10} + \frac{c_n}{10^2} + \cdots] = \frac{1}{9}|\beta_n(z)|c_n.$$

Together with (16) and (17), this gives (15). □

Claim 2. Fix $z \in \gamma_1$. Let g, h be distinct functions in $\mathcal{K}$. Then

(18) $$|g(z) - h(z)| \geq \tfrac{3}{2}|\beta_n(z)|c_n.$$

Proof of Claim.

$$g(z) = \sum_{j=1}^{n} c_j\rho_j\beta_j(z), \qquad h(z) = \sum_{j=1}^{n} c_j\rho_j'\beta_j(z),$$

where ρ_j, ρ_j' are constants $=$ 1 or -1. For some j, $\rho_j \neq \rho_j'$. Let j_0 be the first such j. Then

$$g(z) - h(z) = \pm 2c_{j_0}\beta_{j_0}(z) + \sum_{j=j_0+1}^{n} c_j(\rho_j - \rho_j')\beta_j(z).$$

So

$$|g(z) - h(z)| \geq 2c_{j_0}|\beta_{j_0}(z)| - 2\sum_{j=j_0+1}^{n} c_j|\beta_j(z)|$$

$$\geq 2c_{j_0}|\beta_{j_0}(z)| - 2|\beta_{j_0}(z)|\sum_{j=j_0+1}^{n} c_j$$

$$\geq 2|\beta_{j_0}(z)||c_{j_0} - \sum_{j=j_0+1}^{n} c_j|$$

$$\geq 2|\beta_{j_0}(z)|c_{j_0}(1 - \tfrac{1}{9}) = \tfrac{16}{9}|\beta_{j_0}(z)|c_{j_0}$$

$$\geq \tfrac{3}{2}|\beta_n(z)|c_n,$$

proving our claim. □

Claim 3. Choose $k_0 \in \mathcal{K}$ such that $|f(z_0) - k_0(z_0)| \leq (1/4)|\beta_n(z_0)|c_n$, which is possible by Claim 1. Then we have for all $z \in \gamma_1$

(19) $$|f(z) - k_0(z)| \leq \tfrac{1}{3}|\beta_n(z)|c_n.$$

Proof of Claim. Put $\mathcal{O} = \{z \in \gamma_1 : \text{(19) holds at } z\}$. The $\mathcal{O}$ is an open subset of γ_1, containing z_0. If $\mathcal{O} \neq \gamma_1$, then there is a boundary point p of $\mathcal{O}$ on γ_1. Then

$$|f(p) - k_0(p)| = \tfrac{1}{3}|\beta_n(p)|c_n. \tag{20}$$

By Claim 1, there is some $k_1 \in \mathcal{K}$ such that

$$|f(p) - k_1(p)| \leq \tfrac{1}{4}|\beta_n(p)|c_n. \tag{21}$$

Thus $|k_0(p) - k_1(p)| \leq (7/12)|\beta_n(p)|c_n$. Also, $k_0 \neq k_1$, in view of (20) and (21). This contradicts (18). Hence $\mathcal{O} = \gamma_1$, and so the claim follows. □

For each continuous function u defined on γ_1 that has a jump at z_1, let us write $L^+(u)$ and $L^-(u)$ for the two limits of $u(z)$ as $z \to z_1$ along γ_1. Then, by (19),

$$|L^+(f) - L^+(k_0)| \leq \tfrac{1}{3}|\beta_n(z_1)|c_n$$

and

$$|L^-(f) - L^-(k_0)| \leq \tfrac{1}{3}|\beta_n(z_1)|c_n.$$

Hence

$$|(L^+(f) - L^-(f)) - (L^+(k_0) - L^-(k_0))| \leq \tfrac{2}{3}|\beta_n(z_1)|c_n.$$

But f is continuous at z_1, so the jump of k_0 at z_1 is in modulus less than or equal to $(2/3)|\beta_n(z_1)|c_n$. But $k_0 \in \mathcal{K}$, and so its jump at z_1 is $2|\beta_n(z_1)|c_n$. This is a contradiction.

So f does not exist, and Lemma 24.6 is proved. □

Lemma 24.7. *Suppose that D is an analytic disk contained in X. Then z is a constant on D ("D is a vertical disk").*

Proof. If z is nonconstant on D, then, without loss of generality, D is given by an equation $w = f(z)$, where f is a single-valued analytic function on a domain $\Omega \subseteq \{|z| < 1/2\}$. Then f is continuous on Ω and the graph of f is contained in X. This contradicts Lemma 24.6. So z is constant on D, as desired. □

Fix z_0 and denote by $\pi^{-1}(z_0)$ the fiber over z_0, i.e.,

$$\pi^{-1}(z_0) = \{w \in \mathbb{C} : (z_0, w) \in X\}.$$

Lemma 24.8. *Let K be a connected component of $\pi^{-1}(z_0)$. Then K is a single point.*

Proof. We first assume that $z_0 \neq a_j$ for all j. Fix an integer N. For each ν, choose one of the values of B_ν at z_0 and denote it $B_\nu(z_0)$. For $j = 1, 2, \dots, 2^N$, put

$$w_j = \sum_{\nu=1}^{N} c_\nu \rho_\nu^{(j)} B_\nu(z_0),$$

where $\rho_1^{(j)}, \rho_2^{(j)}, \ldots, \rho_N^{(j)}$ is an N-tuple of $1's$ and $(-1)'s$. Then w_j, $j = 1, 2, \ldots, 2^N$, are the w-coordinates of the points of Σ_N lying over z_0. By a calculation similar to the one in the proof of (18), we find that

$$|w_j - w_k| \geq \tfrac{3}{2}|B_N(z_0)|c_N, \qquad 1 \leq j, k \leq 2^N, \ j \neq k. \tag{22}$$

Consider the closed disks with center w_j, $j = 1, 2, \ldots, 2^N$, and radius $(1/2)|B_N(z_0)|c_N$. Because of (22), these disks are disjoint.

Claim. Fix $b \in \pi^{-1}(z_0)$. Then b belongs to the union of the 2^N disks.

PROOF OF THE CLAIM. By Lemma 24.5 there exists $M > N$ and there exists $(z_0, w') \in \Sigma_M$ such that

$$|b - w'| < \tfrac{1}{9} c_N |B_N(z_0)|.$$

Now $w' = \sum_{\nu=1}^{M} c_\nu \rho_\nu B_\nu(z_0)$, where each $\rho_\nu = 1$ or $= -1$.

Since $(z_0, w') \in \Sigma_M$,

$$w' = \sum_{\nu=1}^{N} c_\nu \rho_\nu B_\nu(z_0) + \sum_{\nu=N+1}^{M} c_\nu \rho_\nu B_\nu(z_0)$$

$$= w_j + \sum_{\nu=N+1}^{M} c_\nu \rho_\nu B_\nu(z_0)$$

for some j, $1 \leq j \leq 2^N$. So

$$|w' - w_j| \leq \sum_{\nu=N+1}^{M} c_\nu |B_\nu(z_0)| \leq \tfrac{1}{9} c_N |B_N(z_0)|.$$

Thus b belongs to the disk with center w_j and radius $(1/2)c_N|B_N(z_0)|$, and hence to the union of these 2^N disks, as claimed.

Since K is a connected component of $\pi^{-1}(z_0)$ and K is contained in the union of the disjoint disks

$$\{|w - w_j| \leq (1/2)c_N|B_N(z_0)|\}, \qquad j = 1, \ldots, 2^N,$$

it follows that K is contained in one of the disks and so diam $K \leq c_N$. This holds of each N. So diam $K = 0$, and therefore Lemma 24.8 is proved in this case.

If $z_0 = a_j$ for some j, then $B_N(z_0) = 0$ for $N > j$. Hence $\pi^{-1}(z_0)$ is finite, and so again each component of $\pi^{-1}(z_0)$ is a single point. □

It follows from Lemma 24.8 that X contains no disk on which z is constant (no "vertical disk").

Lemma 24.9. *Put*

$$Y = X \cap \{|z| = \tfrac{1}{2}\}.$$

Then $X = \hat{Y}$.

PROOF. Since X is polynomially convex, $\hat{Y} \subseteq X$.

Now fix $(z, w) \in X$. Choose a sequence $\{(z, w_k)\}$ converging to (z, w) such that $(z, w_k) \in \Sigma_k$ for each k. For each k put $\partial\Sigma_k = \Sigma_k \cap \{|z| = 1/2\}$.

Let Q be a polynomial on $\mathbb{C}^2$. By the maximum principle of Σ_k, for each k,

$$|Q(z, w_k)| \leq |Q(z'_k, w'_k)|,$$

where $(z'_k, w'_k) \in \partial\Sigma_k$.

Let (z^*, w^*) be an accumulation point of the sequence $\{(z'_k, w'_k)\}$. Fix n_0. For $k > n_0$, $(z'_k, w'_k) \in \{|P_k| \leq \epsilon_k\} \subseteq \{|P_{n_0}| \leq \epsilon_{n_0}\}$. Letting $k \to \infty$, we get

$$(z^*, w^*) \in \{|P_{n_0}| \leq \epsilon_{n_0}\}.$$

Since this holds for each n_0, $(z^*, w^*) \in X$. Also, $|z^*| = 1/2$. Further, by letting $k \to \infty$, we get

$$|Q(z, w)| \leq |Q(z^*, w^*)|.$$

So $|Q(z, w)| \leq \max |Q|$ over $X \cap \{|z| = 1/2\} = Y$. Thus $(z, w) \in \hat{Y}$, and so $X \subseteq \hat{Y}$.

It follows that $X = \hat{Y}$. □

In view of Lemmas 24.7 , 24.8, and 24.9, $\hat{Y}$ contains no analytic disk. Also, $\pi(Y) = \{|z| = 1/2\}$ and $\pi(\hat{Y}) = \{|z| \leq 1/2\}$.

Theorem 24.3 is proved. □

EXAMPLE 24.4. We now discuss an example of Ahern and Rudin [AR] of a totally real 3-sphere Σ in $\mathbb{C}^3$. We refer to Item 9 of Chapter 25 for the significance of this type of example. We recall that "totally real" means that the tangent space at each point contains no complex subspace of positive dimension. Let $S^3 = \{(z, w) \in \mathbb{C}^2 : z\overline{z} + w\overline{w} = 1\}$ and let σ be a smooth complex-valued function defined on a neighborhood of S^3. Let Σ be the 3-sphere in $\mathbb{C}^3$ that is the image of S^3 under the embedding $E : S^3 \to \mathbb{C}^3$ given by $E(z, w) = (z, w, \sigma(z, w))$; i.e., Σ is the graph of $\sigma | S^3$. Set

$$L = w\frac{\partial}{\partial\overline{z}} - z\frac{\partial}{\partial\overline{w}},$$

the tangential Cauchy–Riemann operator on S^3.

Proposition 24.10. *Σ is totally real if and only if $L\sigma \neq 0$ at every point of S^3.*

PROOF. Fix $(a, b) \in S^3$. Then the complex line tangent ℓ to S^3 at (a, b) can be parameterized by

$$\lambda \to (a + \overline{b}\lambda, b - \overline{a}\lambda).$$

It is straightforward to check that

$$\sigma(a + \overline{b}\lambda, b - \overline{a}\lambda) = \sigma(a, b) + (\overline{L}\sigma)(a, b)\lambda + (L\sigma)(a, b)\overline{\lambda} + o(|\lambda|),$$

as $\lambda \to 0$, where $\overline{L} = \overline{w}\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial w}$; indeed, it suffices to show that $\frac{\partial}{\partial \lambda}$ and $\frac{\partial}{\partial \overline{\lambda}}$ of both sides agree at $\lambda = 0$. The orthogonal projection $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ takes a complex tangent to Σ at $(a, b, \sigma(a, b))$ to the complex line ℓ, and so the only possible complex tangent line to Σ at $(a, b, \sigma(a, b))$ is a graph over ℓ of the first-order terms of $\sigma(a + \overline{b}\lambda, b - \overline{a}\lambda)$, viewed as a function of λ. This is a complex line (no $\overline{\lambda}$ term!) if and only if $L\sigma(a, b) = 0$. □

We shall need the following.

Lemma 24.11. *Let Ω be a bounded domain in $\mathbb{C}$ such that $0 \in b\Omega$ and such that Ω is disjoint from the negative real axis. Let $\zeta_0 \in \Omega$. For $0 < r < |\zeta_0|$, let $\Omega_r = \{\zeta \in \Omega : |\zeta| > r\}$ and let $\alpha_r = \{\zeta \in b\Omega_r : |\zeta| = r\}$; assume that Ω_r is connected. Then there exists $C > 0$ (depending on ζ_0 but not on r) such that the harmonic measure of α_r with respect to the point ζ_0 and the domain Ω_r is $\leq C\sqrt{r}$.*

Proof. Denote by $\sqrt{\zeta}$ the principal value of the square root on the plane cut by the negative real axis. In the right half-plane define a nonnegative harmonic function

$$H(z) = \frac{2}{\pi}\arg\left(\frac{z + i\sqrt{r}}{z - i\sqrt{r}}\right).$$

We have $H(z) \equiv 1$ if $|z| = \sqrt{r}$. On Ω_r, define a nonnegative harmonic function $h(\zeta) = H(\sqrt{\zeta})$. Since $h \geq 0$ on Ω_r and $h \equiv 1$ on α_r, we get:

$$h(\zeta_0) \geq \text{harmonic meas}(\alpha_r).$$

Finally, we use the estimate

$$h(\zeta_0) = \frac{2}{\pi}\arg\left(\frac{\sqrt{\zeta_0} + i\sqrt{r}}{\sqrt{\zeta_0} - i\sqrt{r}}\right) \leq C\sqrt{r}$$

for some $C > 0$. □

We now make a special choice of σ. For $(z, w) \in \mathbb{C}^2$ set $\sigma(z, w) = \overline{zw}(w\overline{w} + iz\overline{z})$. It is straightforward to check that $L\sigma \neq 0$ at every point of S^3, and so we obtain the totally real 3-sphere in $\mathbb{C}^3$ that we seek as the 3-sphere Σ, which is the graph of the function σ on S^3.

We know, say by Browder's Theorem 15.8, that Σ is not polynomially convex. To conclude this example, we shall determine the polynomially convex hull of Σ. For this we shall use a method of Anderson [An] and Wermer [We8]. Let $F(z) = z_1z_2z_3$, a polynomial in $\mathbb{C}^3$, where $z = (z_1, z_2, z_3)$. Then, for $z \in \Sigma$, we have $F(z) = z_1z_2\sigma(z_1, z_2) = |z_1|^2|z_2|^2(|z_2|^2 + i|z_1|^2)$. Thus the set $F(\Sigma)$ is a curve Γ in the plane parameterized by $\gamma : [0, 1] \to \mathbb{C}$, given by $\gamma(t) = t^2(1 - t^2)(1 - t^2 + it^2)$. We note that $|\gamma(t)| \leq t^2(1 - t^2)$ and hence $|\zeta| \leq 1$ on Γ. Since γ is one-to-one except that $\gamma(0) = 0 = \gamma(1)$, Γ is a Jordan curve through the origin bounding a domain Ω. Moreover, Γ is smooth except at the origin, where there is a cusp consisting of two curves that meet with an internal angle of $\pi/2$; this implies that the harmonic measure on Γ for points in Ω is of the form $K\,ds$, where ds denotes arc length and K is continuous on Γ.

Let τ be the real-valued function on $\Gamma \setminus \{0\}$ that is the inverse of $\gamma|_{(0,1)}$. Then, for $z \in \Sigma$, $\gamma(|z_1|) = F(z)$ and so the fiber $(F|\Sigma)^{-1}(\zeta)$ for $\zeta \neq 0 \in \Gamma$ is the torus $\{(z_1, z_2, z_3) : |z_1| = \tau(\zeta), |z_2| = \sqrt{1-\tau(\zeta)^2},$ and $z_3 = \zeta/(z_1 z_2)\}$.

Lemma 24.12.
(a) *For $\zeta \neq 0 \in \Gamma$, the torus $(F|\Sigma)^{-1}(\zeta)$ is polynomially convex.*
(b) $F(\hat{\Sigma}) = \overline{\Omega}$.

PROOF.
(a) It follows easily that the polynomials are dense in the space of all continuous functions on $(F|\Sigma)^{-1}(\zeta)$ since $\overline{z}_1$ and $\overline{z}_2$ are in the closure of the polynomials on $(F|\Sigma)^{-1}(\zeta)$. To see this, one need only write $\overline{z}_1 = \tau(\zeta)^2/z_1 = \tau(\zeta)^2 z_3 z_2/\zeta$ on Σ and similarly for $\overline{z}_2$.
(b) It suffices to show that $F(\hat{\Sigma}) \supset \Omega$—the rest is clear. Suppose that this is not the case. Then $F(\hat{\Sigma})$ is disjoint from Ω, and so $F(\hat{\Sigma}) = \Gamma$. Since every point of Γ is a peak point of $P(\overline{\Omega})$, it follows that $(F|\hat{\Sigma})^{-1}(\zeta) = \widehat{(F|\Sigma)^{-1}}(\zeta) = (F|\Sigma)^{-1}(\zeta)$ for all $\zeta \neq 0 \in \Gamma$. Hence $\hat{\Sigma} \setminus \Sigma \subseteq (F|\Sigma)^{-1}(\{0\})$. Hence $z_1 z_2 \equiv 0$ on $\hat{\Sigma} \setminus \Sigma$, and in particular the projection of Σ to $\mathbb{C}^2$ does not cover the unit ball in $\mathbb{C}^2$. This contradicts Theorem 22.6.

$\square$

For $\zeta \in \Omega$ we define

$$Z_i(\zeta) = \max\{|z_i| : z \in \hat{\Sigma} \text{ and } F(z) = \zeta\}, \qquad i = 1, 2, 3.$$

By a previous result we know that $\log Z_i$ is subharmonic on Ω. We need to examine the boundary behavior of the Z_i. By Lemma 24.12, $(F|\hat{\Sigma})^{-1}(\zeta) = (F|\Sigma)^{-1}(\zeta)$ for all $\zeta \neq 0 \in \Gamma$. Hence, since $\hat{\Sigma}$ is compact, it follows that the boundary values of the Z_i are given by: $Z_1(\zeta) = \tau(\zeta)$, $Z_2(\zeta) = \sqrt{1-\tau(\zeta)^2}$, and $Z_3(\zeta) = |\zeta|/(\tau(\zeta)\sqrt{1-\tau(\zeta)^2})$ for all $\zeta \neq 0 \in \Gamma$. Consider harmonic functions U_i in Ω with boundary values $\log Z_i$, for $i = 1, 2, 3$. Since the functions $\log Z_i$ are not bounded on Γ, the existence of the U_i needs justification. It follows from the estimate $\log Z_i \geq \log|\zeta| - A$ on Γ, for $i = 1, 2, 3$ (see the proof of Lemma 24.13), and the remark above about harmonic measure on Γ that the U_i can be defined by $U_i(\zeta) = \int \log Z_i(\lambda)\, d\mu_\zeta(\lambda)$ for all $\zeta \in \Omega$, where μ_ζ is harmonic measure for ζ on Γ. In particular, $U_3(\zeta) = \log|\zeta| - U_1(\zeta) - U_2(\zeta)$ for all $\zeta \in \Omega$.

Since we are dealing with unbounded functions, the following inequality is not a direct consequence of the maximum principle.

Lemma 24.13. $\log Z_i(\zeta) \leq U_i(\zeta)$ *for all* $\zeta \in \Omega$, $i = 1, 2, 3$.

PROOF. Fix $\zeta_0 \in \Omega$ and let $0 < r < |\zeta_0|$. Let μ^r be harmonic measure for ζ_0 on $b\Omega_r$. In Ω, $\log Z_1$ is a subharmonic function bounded above by M. Since $\log Z_1$

is subharmonic and has continuous boundary values in Ω_r , we have

$$\log Z_1(\zeta_0) \leq \int_{b\Omega_r} \log Z_1(\lambda) d\mu^r(\lambda).$$

Splitting $b\Omega_r$ into the disjoint union of α_r (recall Lemma 24.11) and β_r, we get

$$\log Z_1(\zeta_0) \leq \int_{\beta_r} U_1(\lambda) d\mu^r(\lambda) + MC\sqrt{r}. \tag{23}$$

For $\lambda \in \alpha_r$, we have

$$U_1(\lambda) = \int_{\Gamma} \log \tau(\zeta) d\mu_\lambda(\zeta).$$

From the estimate $|\zeta| = |\gamma(\tau(\zeta))| \leq \tau(\zeta)^2(1 - \tau(\zeta)^2)$ for $\zeta \in \Gamma \setminus \{0\}$, we have $\log \tau \geq \log|\zeta|$; hence $U_1(\lambda) \geq \log r$ for $\lambda \in \alpha_r$. Therefore,

$$U_1(\zeta_0) = \int_{\beta_r} U_1 d\mu^r + \int_{\alpha_r} U_1 d\mu^r \geq \int_{\beta_r} U_1 d\mu^r + \log r \cdot C\sqrt{r}. \tag{24}$$

Now, letting $r \to 0$, it follows from (23) and (24) that $\log Z_1(\zeta_0) \leq U_1(\zeta_0)$. An analogous argument gives the statement for $i = 2, 3$. (For $i = 3$, one uses the estimate $\log(|\zeta|/(\tau(\zeta)\sqrt{1 - \tau(\zeta)^2})) \geq 1/2 \log|\zeta| - A$ in place of $\log \tau \geq \log|\zeta|$.) □

For $\zeta \in \Omega$, we have, since $\zeta = F(z)$ for some $z \in \hat{\Sigma}$, that $|\zeta| = |z_1z_2z_3| \leq |Z_1(\zeta)Z_2(\zeta)Z_3(\zeta)|$ and therefore $\log|\zeta| = \log|z_1| + \log|z_2| + \log|z_3| \leq \log Z_1(\zeta) + \log Z_2(\zeta) + \log Z_3(\zeta) \leq U_1(\zeta) + U_2(\zeta) + U_3(\zeta) \equiv \log|\zeta|$. We conclude that $U_i \equiv \log Z_i$ on Ω and that $Z_i(z_1z_2z_3) \equiv |z_i|$.

Let V_i be a harmonic conjugate of U_i in Ω and set $\phi_i = e^{U_i + \sqrt{-1}V_i}$, for $i = 1, 2$. Put $\phi(\zeta) = (\phi_1(\zeta), \phi_2(\zeta), \zeta/(\phi_1(\zeta)\phi_2(\zeta)))$, an analytic map from Ω to $\mathbb{C}^3$. We claim that $\phi(\Omega) \subseteq \hat{\Sigma}$. First note that, for fixed $\theta_1, \theta_2 \in \mathbb{R}$, Σ is invariant under the map $(z_1, z_2, z_3) \mapsto (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{-i(\theta_1+\theta_2)}z_3)$, since $\sigma(e^{i\theta_1}z_1, e^{i\theta_2}z_2) = e^{-i(\theta_1+\theta_2)}\sigma(z_1, z_2)$. Therefore, $\hat{\Sigma}$ is invariant under the same maps. Let $\zeta \in \Omega$. Hence there exists $(z_1, z_2, z_3) \in \hat{\Sigma}$ such that $F((z_1, z_2, z_3)) = \zeta$ and $Z_i(z_1z_2z_3) = |z_i|, i = 1, 2, 3$. Hence $|\phi_i(\zeta)| = e^{U_i(\zeta)} = Z_i(\zeta) = |z_i|$, i.e., $\phi_j(\zeta) = e^{i\theta_j}z_j$, $j = 1, 2$. By the invariance of $\hat{\Sigma}$ we conclude that $\phi(\zeta) \in \hat{\Sigma}$.

Now set $\phi_{\theta_1,\theta_2}(\zeta) = (e^{i\theta_1}\phi_1(\zeta), e^{i\theta_2}\phi_2(\zeta), e^{-i(\theta_1+\theta_2)}\zeta/(\phi_1(\zeta)\phi_2(\zeta)))$, for $(\theta_1, \theta_2) \in [0, 2\pi) \times [0, 2\pi)$. The argument of the last paragraph shows that $\phi_{\theta_1,\theta_2}(\Omega) \subseteq \hat{\Sigma}$. Conversely, the same argument shows that if $z = (z_1, z_2, z_3) \in \hat{\Sigma}$ and $F(z) \in \Omega$, then there exists ϕ_{θ_1,θ_2} such that $z = \phi_{\theta_1,\theta_2}(F(z))$. Thus we have shown that $\hat{\Sigma} \cap F^{-1}(\Omega)$ is a disjoint union of the analytic disks $\phi_{\theta_1,\theta_2}(\Omega)$. In view of Lemma 24.12 and the comments preceding it, it remains only to determine the set $\hat{\Sigma} \cap F^{-1}(\{0\})$. Since $\Sigma \cap F^{-1}(\{0\})$ is the union of the two circles $\{(z_1, z_2, z_3) : |z_1| = 1, z_2 = z_3 = 0\}$ and $\{(z_1, z_2, z_3) : |z_2| = 1, z_1 = z_3 = 0\}$, it follows that $\hat{\Sigma} \cap F^{-1}(\{0\})$ is the union of the two disks $\{(z_1, z_2, z_3) : |z_1| \leq 1, z_2 = z_3 = 0\}$ and $\{(z_1, z_2, z_3) : |z_2| \leq 1, z_1 = z_3 = 0\}$. This completes the construction of the hull of Σ. We have shown that $\hat{\Sigma} \setminus \Sigma$ is the disjoint union of the 2-parameter family of disks $\phi_{\theta_1,\theta_2}(\Omega)$ and the two coordinate disks. We mention here without proof that Ahern and Rudin [AR] have shown further that $\hat{\Sigma}$ is a graph over the closed unit ball in $\mathbb{C}^2$.

We note that the disks ϕ_{θ_1,θ_2} are not smooth on the unit circle; they are examples of H^∞ disks. H^∞ *disks* are bounded analytic mappings whose boundary values exist only a.e. on the unit circle. This is in contrast to *analytic disks*, which extend smoothly to the unit circle. More precisely, *except at one point of the unit circle*, the ϕ_{θ_1,θ_2} are smooth on the circle and with boundary values in Σ. It turns out that such special nonconstant H^∞ disks always exist for n-dimensional totally real submanifolds of $\mathbb{C}^n$—see Chapter 25.

EXAMPLE 24.5. We next give an example of an arc in $\mathbb{C}^3$ that is not polynomially convex. In this connection recall that we have shown in Theorem 12.4 that a smooth arc is polynomially convex! If γ is an arc in the plane, denote by $\mathfrak{A}_\gamma$ the algebra of functions continuous on the Riemann sphere S^2 and analytic on $S^2 \setminus \gamma$.

Lemma 24.14. *If γ has positive plane measure, then $\mathfrak{A}_\gamma$ contains three functions that separate the points on S^2.*

Remark. To obtain an arc having positive plane measure, one can proceed as follows: Choose a compact totally disconnected set E on the real line, having positive linear measure. Then $E \times E$ is a compact, totally disconnected subset of $\mathbb{R}^2$ having positive planar measure. Through every compact totally disconnected subset of the plane an arc may be passed, as was shown by F. Riesz [Rie] ; γ can be such an arc. The first example of an arc of positive planar measure was found by Osgood in 1903 by a different method.

PROOF. Put

$$F(\zeta) = \int_\gamma \frac{dx\,dy}{z - \zeta}.$$

$F(\zeta) \to 0$ as $\zeta \to \infty$ and $\lim_{\zeta\to\infty} \zeta \cdot F(\zeta) \neq 0$. Hence F is not a constant. Fix $\zeta_0 \in \gamma$. The integral defining $F(\zeta_0)$ converges absolutely. (Why?) We claim that F is continuous at ζ_0. For this put

$$g(z) = \begin{cases} 1/z & \text{for } |z| < R \\ 0 & \text{for } |z| \geq R \end{cases}$$

for R some large number. Then $g \in L^1(\mathbb{R}^2)$:

$$|F(\zeta) - F(\zeta_0)| \leq \int_\gamma \left| \frac{1}{z-\zeta} - \frac{1}{z-\zeta_0} \right| dx\,dy$$

$$= \int_\gamma |g(z-\zeta) - g(z-\zeta_0)| dx\,dy \to 0$$

as $\zeta \to \zeta_0$, since $g \in L^1(\mathbb{R}^2)$. Hence the claim is established. Thus $F \in C(S^2)$, and, since F evidently is analytic on $S^2 \setminus \gamma$, $F \in \mathfrak{A}_\gamma$.

Now fix $a, b \in S^2 \setminus \gamma$ with $F(a) \neq F(b)$. Then $F_2, F_3 \in \mathfrak{A}_\gamma$, where

$$F_2(z) = \frac{F(z) - F(a)}{z - a}, \qquad F_3(z) = \frac{F(z) - F(b)}{z - b}.$$

Fix distinct points $z_1, z_2 \in S^2$. It is easily checked that if $F(z_1) = F(z_2)$, then either F_2 or F_3 separates z_1 and z_2. Hence F, F_2, and F_3 together separate points on S^2. □

We now define an arc J_0 in $\mathbb{C}^3$ as the image of a given plane curve γ having positive planar measure under the map $\zeta \longmapsto (F(\zeta), F_2(\zeta), F_3(\zeta))$.

Theorem 24.15. *J_0 is not polynomially convex in $\mathbb{C}^3$. Hence $P(J_0) \neq C(J_0)$.*

PROOF. Fix $\zeta_0 \in S^2 \setminus \gamma$. Then $x^0 = (F(\zeta_0), F_2(\zeta_0), F_3(\zeta_0)) \notin J_0$. Yet, if P is any polynomial in $\mathbb{C}^3$,

$$|P(x^0)| \leq \max_{J_0} |P|.$$

Indeed, $f = P(F, F_2, F_3) \in \mathfrak{A}_\gamma$, and so, by the maximum principle,

$$|f(\zeta_0)| \leq \max_{\gamma} |f|,$$

as asserted. Hence $x^0 \in \hat{J}_0 \setminus J_0$, and we are done. □

EXERCISE 24.2. If ϕ is a nonconstant element of $P(J_0)$, then $\phi(J_0)$ is a Peano curve in $\mathbb{C}$; i.e., $\phi(J_0)$ contains interior points. In particular, the coordinate projections of J_0, $z_k(J_0)$, are points or Peano curves. [Hint: Apply the argument principle to show that $f(S^2) = f(J_0)$ for all $f \in \mathfrak{A}_\gamma$—use the fact that γ is an arc.]

NOTES

Example 24.1, which is a variant of Stolzenberg's example, is given in Wermer [We9]. Example 24.2 is given in B. Coles's thesis [Col]. Example 24.3 is given by Wermer in [We11]. Example 24.4 is due to Ahern and Rudin in [AR] and to J. Anderson [An]. The example of a nonpolynomially convex arc, Example 24.5, is due to Wermer in [We2]. Such an arc in $\mathbb{C}^2$ was constructed by Rudin in [Ru2].

The phenomenon of a small set in $\mathbb{C}^n$ having a large hull was exhibited by Vituškin [V]. He constructed a compact totally disconnected set in $\mathbb{C}^2$ whose polynomial hull contains the bidisk.

25

Historical Comments and Recent Developments

1 Introduction

We shall discuss some historical background, including some recent applications. We also shall supply some references for Chapters 11, 12, 20, and 23.

We begin with the following.

Definition 25.1. Let Ω be a domain in $\mathbb{C}^n$ and let E be a relatively closed subset of Ω. E is called *pseudoconcave in* Ω if the open set $\Omega \setminus E$ is pseudoconvex.

Note. The term "pseudoconcave" first appeared in Nishino's paper [Ni].

EXAMPLE 25.1. Let Ω be the cylindrical domain $\{|z| < 1\} \times \mathbb{C}$ in $\mathbb{C}^2$. Chose an analytic function f on $\{|z| < 1\}$ and let E be the graph of f, i.e., $E = \{(z, f(z)) : |z| < 1\}$. Then E is pseudoconcave in Ω.

PROOF. Put $\psi(z, w) = -\log |w - f(z)|$ for $(z, w) \in \Omega \setminus E$. Since $(w - f(z))^{-1}$ is analytic on $\Omega \setminus E$, clearly ψ is an exhaustion function for $\Omega \setminus E$. $\square$

EXERCISE 25.1. Choose Ω as in the preceding example. Let $a_1, a_2, \ldots, a_n$ be analytic functions on $\{|z| < 1\}$ and let E be the subset of Ω given by

$$w^n + a_1(z)w^{n-1} + \cdots + a_n(z) = 0.$$

Then E is pseudoconcave in Ω.

2 Hartogs' Theorem

In his fundamental paper [Ha] in 1909, F. Hartogs proved results including the following.

Theorem 25.1. *Let E be a pseudoconcave set in $\{|z| < 1\} \times \mathbb{C}$ in $\mathbb{C}^2$ that lies one-sheeted over $\{|z| < 1\}$, in the sense that each complex line $\{z = z_0\}$ with*

$|z_0| < 1$ *meets* E *exactly once. Then* E *is the graph of an analytic function on* $\{|z| < 1\}$.

Hartogs also proved in [Ha] the analogous result for the case that E lies finite-sheeted over $\{|z| < 1\}$.

In the short paper [Ok1] in 1934, which contained statements of results but no proofs, K. Oka developed the theory of pseudoconcave sets, extending Hartogs' work. In 1962, T. Nishino in [Ni] gave an exposition of the theory, with detailed proofs. In particular, Oka and Nishino gave results for pseudoconcave sets which are models for many of the results that we proved for maximum modulus algebras in Chapter 11.

3 Maximum Modulus Algebras

Maximum modulus algebras (A, X, Ω, p), with X a plane region and p the identity function, occurred first in Rudin's paper [Ru1] in 1953. In particular, Rudin proved Theorem 10.3 there.

Rossi's Local Maximum Modulus Principle, which is given as Theorem 9.3 above, vastly increased the list of examples of maximum modulus algebras. (See Theorem 11.9 above.)

In 1980, in [We12] one of us proved

Theorem 25.2. *Let* X *be a pseudoconcave set contained in the open bidisk* $\{|z| < 1, |w| < 1\} \subseteq \mathbb{C}^2$. *Denote by* A *the algebra of polynomials in* z *and* w *restricted to* X. *Putting* $D = \{|z| < 1\}$ *and* $\pi : (z, w) \mapsto z$, *then* (A, X, D, π) *is a maximum modulus algebra.*

In 1981, in [Sl1], Z. Slodkowski introduced the concept of an *analytic set-valued function*. Consider a map $\Phi : \lambda \mapsto K_\lambda$ defined on a plane region with values that are compact subsets of $\mathbb{C}$. We say that Φ is *upper semi-continuous* at λ_0 if, given any neighborhood N of K_{λ_0} in $\mathbb{C}$, there exists an $\epsilon > 0$ such that $K_\lambda \subseteq N$ if $|\lambda - \lambda_0| < \epsilon$.

Definition 25.2. The set-valued map $\Phi : \lambda \mapsto K_\lambda$, with K_λ a compact subset of $\mathbb{C}$ for each $\lambda \in G$, is *analytic* if:

(1) Φ is upper semi-continuous, and (2) The set of all (λ, w) with $\lambda \in G$ and $w \in \mathbb{C} \setminus K_\lambda$ is a pseudoconvex subset of $\mathbb{C}^2$.

Note. We could also express condition (2) by saying that the graph of Φ consisting of all points (λ, w) with $\lambda \in G$ and $w \in K_\lambda$ is pseudoconcave in $G \times \mathbb{C}$.

Slodkowski showed in Theorem 2.1 of [Sl1] that, if X is a relatively closed set contained in a cylinder domain $G \times \mathbb{C} \subseteq \mathbb{C}^2$, A is the restriction to X of the algebra of polynomials in z and w, and π is the projection $(z, w) \mapsto z$, then (A, X, G, π) is a maximum modulus algebra if and only if X is pseudoconcave in $G \times \mathbb{C}$.

Note. The definition of maximum modulus algebra in [Sl1] is somewhat stronger than the definition we have given in Chapter 11. For details see [Sl1].

Slodkowski's paper [Sl1] contains a number of interesting results relating analytic set-valued functions to operator theory and to the study of uniform algebras. An exposition of these relationships, and related questions, is given by B. Aupetit in [Au1] and [Au2].

An expository article on maximum modulus algebras is given by one of us in [We13]. Further work in this field is to be found in the book *Uniform Frechet Algebras* [Go] by H. Goldman, Chapters 15 and 16. Proofs of Theorem 11.7 were given by Slodkowski in [Sl2] and Senichkin in [Sen]. See also Kumagai [Kum].

4 Curve Theory

In Chapter 12 we studied the problem of finding the hull of a given curve γ in $\mathbb{C}^n$. The case of a real-analytic curve was treated in the 1950s by one of us in the papers [We3], [We4], [We5], and [We6]. The principal tool in these papers was the Cauchy transform. An elegant treatment of this case and applications to the study of the algebra of bounded analytic functions on Riemann surfaces was given in [Ro2].

In two very influential papers, [Bi3] and [Bi2], Errett Bishop gave an abstract Banach algebra approach to the problem of finding hulls of curves. In particular, Bishop he proved a version of our Theorem 11.8 for the case of Banach algebras in [Bi3].

Based in part on Bishop's work, G. Stolzenberg solved the problem for $\mathcal{C}^1$-smooth curves in [St2]. The case when γ is merely rectifiable was treated by Alexander in [Al1].

Independent of the study of algebras of functions, B. Aupetit in [Au3] applied the theory of subharmonic functions to problems in the spectral theory of operators. Aupetit and Wermer in [AuWe] gave a new proof and generalization of Bishop's result in [Bi3], by adapting the methods used in [Au3].

An independent proof of the result in [AuWe] was given by Senichkin in [Sen].

5 Boundaries of Complex Manifolds

Given a k-dimensional manifold X in $\mathbb{C}^n$, identifying the polynomial hull in the case $k > 1$ turned out to be a much harder problem than in the case $k = 1$.

The first major result was found by A. Browder in [Bro1] in the case $k = n$. Let X be a compact orientable n-manifold in $\mathbb{C}^n$; Browder shows that $\hat{X}$ is always larger than X.

EXERCISE 25.2. Why is this true when $k = n = 1$?

In [Al2], Alexander obtained the stronger result that, if X is as in Browder's situation, then the closure of $\hat{X} \setminus X$ contains X, so $\hat{X} \setminus X$ is "large."

Let X be a k-dimensional smooth oriented manifold in $\mathbb{C}^n$, where k is an odd integer. If X is the boundary of a complex manifold Σ with $\Sigma \cup X$ compact, then $\Sigma \subseteq \hat{X}$. So we may ask: Given X, when does such a Σ exist?

The solution was found in 1975 by R. Harvey and Blaine Lawson in their fundamental paper [HarL2] and developed in [Har]. To obtain a tractable problem, one allows Σ to have singularities and thus seeks an analytic variety Σ with boundary X, rather than a manifold. We have sketched a proof of the result of [HarL2] in Chapter 19 for X in $\mathbb{C}^3$.

One may ask a related question: Given a closed curve in the complex projective plane $\mathbb{CP}^2$, when does there exist an analytic variety in $\mathbb{CP}^2$ with boundary γ? This problem was solved by P. Dolbeault and G. Henkin in [DHe].

6 Sets Over the Circle

Let X be a compact set in $\mathbb{C}^n$ lying over the unit circle. Suppose that under the projection $(\lambda, w_1, \ldots, w_{n-1}) \mapsto \lambda$, $\hat{X}$ covers some point in the open disk $\{|\lambda| < 1\}$ and hence covers every point. We are interested in discovering all analytic disks, if any, contained in $\hat{X} \setminus X$.

Theorem 20.2 tells us that if the fiber X_λ with $\lambda \in \Gamma$ is a convex set, then $\hat{X} \setminus X$ is the union of a family of analytic disks, each of which is moreover a graph over $\{|\lambda| < 1\}$. Theorem 20.2 was proved independently by Alexander and Wermer [AW] for $n = 2$ and by Slodkowski [Sl3] for arbitrary n.

Forstnerič showed in [Fo1] that the hypothesis "X_λ is convex for all λ" could be replaced by the hypothesis "X_λ is a simply connected Jordan domain varying smoothly with $\lambda \in \Gamma$, such that $0 \in \mathrm{int}(X_\lambda)$, for all λ," with the same conclusion as in Theorem 20.2. The following stronger result was proved by Slodkowski in [Sl4], and a closely related result was proved by Helton and Marshall in [HeltM]:

Theorem 25.3. *Assume that each fiber X_λ, $\lambda \in \Gamma$, is connected and simply connected. Then $\hat{X} \setminus X$ is a union of analytic graphs over* $\{|\lambda| < 1\}$.

What if the fibers X_λ are allowed to be disconnected? We saw in Chapter 24, Theorem 24.3, that $\hat{X} \setminus X$ may fail to contain any analytic disk, so no extension of Theorem 25.3 to arbitrary sets over the circle is possible.

A number of interesting applications have been found for results concerning polynomial hulls of sets over the circle. We shall write D for the open unit disk.

(i) *Convex domains in* $\mathbb{C}^n$. Let W be a smoothly bounded convex domain in $\mathbb{C}^n$. In [Lem], Lempert constructed a special homeomorphism Φ of W onto the unit ball in $\mathbb{C}^n$, which can be viewed as an analogue of the Riemann map in the case $n = 1$. The construction of Φ is based on certain maps of D into

W, called *extremal*: Given $a \in W$ and $\xi \in \mathbb{C}^n \setminus \{0\}$, an analytic map f of D into W is called extremal with respect to a, ξ if $f(0) = a$, $f'(0) = \lambda\xi$, where $\lambda > 0$, such that, for every analytic map g of D into W with $g(0) = a$, $g'(0) = \mu\xi$ with $\mu > 0$, we have $\lambda \geq \mu$.

It is shown that, given a, ξ, there exists a unique such corresponding extremal map. In [Sl5], Slodkowski gives a construction of Lempert's map Φ by using properties of polynomial hulls of sets over the circle.

(ii) *Corona Theorem*. Carleson's Corona Theorem [Carl2] states that if $f_1, \ldots, f_n$ are bounded analytic functions on D such that there exists $\delta > 0$ with $\sum_{j=1}^{n} |f_j(z)| \geq \delta$ for all $z \in D$, then there exist bounded analytic functions $g_1, \ldots, g_n$ on D satisfying

$$\sum_{j=1}^{n} f_j g_j = 1$$

on D. In [BR], Berndtsson and Ransford gave a geometric proof of the Corona Theorem in the case $n = 2$, basing themselves on the existence of analytic graphs in the polynomial hulls of certain sets in $\mathbb{C}^2$ lying over the circle, as well as results on analytic set-valued functions in [Sl1]. In [Sl6], Slodkowski gave a related proof of the Corona Theorem for arbitrary n.

(iii) *Holomorphic motions*. Let E be a subset of $\mathbb{C}$. A *holomorphic motion* of E in $\mathbb{C}$, parametrized by D, is a map $F : D \times E$ into $\mathbb{C}$ such that:
(a) For fixed $w \in E$, $z \mapsto f(z, w)$ is holomorphic on D.
(b) If $w_1 \neq w_2$, then $f(z, w_1) \neq f(z, w_2)$ for all z in D.
(c) $f(0, w) = w$ for all $w \in E$.
In this "motion," time is the complex variable z.

Extending the earlier work of Sullivan and Thurston [SuT], Slodkowski shows in [Sl7] that a holomorphic motion of an arbitrary subset E of $\mathbb{C}$ can be extended to a holomorphic motion of the full complex plane. As in earlier applications, his proof makes use of results about the structure of polynomial hulls of sets lying over the circle.

(iv) *H^∞ control theory*. A branch of modern engineering known as "H^∞ control theory" leads to mathematical problems of which a simple example is this: for each λ on the unit circle Γ, specify a closed disk Y_λ in $\mathbb{C}$. Find all bounded analytic functions f on the unit disk such that for almost all $\lambda \in \Gamma$, $f(\lambda) \in Y_\lambda$. In view of Theorem 20.2 in Chapter 20, this problem is closely related to finding the polynomial hulls of sets lying over the unit circle. For references, see J. W. Helton [Helt1], [Helt2] as well as the references given therein.

7 Sets with Disk Fibers

Let X be a compact set in $\mathbb{C}^2$ lying over the unit circle Γ such that each fiber X_λ is a closed disk. We write

$$X_\lambda = \{w \in \mathbb{C} : |w - \alpha(\lambda)| \leq R(\lambda)\}$$

for $\lambda \in \Gamma$, where we assume that α is a continuous complex-valued function on Γ satisfying $|\alpha(\lambda)| \leq R(\lambda)$ for all $\lambda \in \Gamma$ and R is a smooth function with values greater than zero. Under these assumptions, it is shown in [AW] that if there exists b with $|b| < 1$ such that $\hat{X}_b$ contains more than one point, then there exists a function Φ of λ and w such that

$$\hat{X} \cap \{|\lambda| < 1\} = \{(\lambda, w) : |\lambda| < 1 \text{ and } |\Phi(\lambda, w)| \leq 1\},$$

and there exist analytic functions A, B, C, D on $|\{\lambda| < 1\}$ such that

$$\Phi(\lambda, w) = \frac{A(\lambda)w + B(\lambda)}{C(\lambda)w + D(\lambda)}, \qquad |\lambda| < 1, |w| < \infty.$$

In the special case where the center function α is a rational function satisfying hypotheses (20.10a) and (20.10b) and $R \equiv 1$, Theorem 20.5 gave an explicit construction of $\hat{X}$. The above-mentioned result is based on the classical result of Adamyan, Arov, and Krein [AAK], which solves the following problem: Give a function h_0 in $L^\infty(\Gamma)$ to describe the totality of functions

$$h = h_0 + \phi, \qquad \phi \in H^\infty$$

such that $||h|| \leq 1$. A proof of the result in [AAK] is also given in Garnett [Ga] and related work is found in Quiggin [Q]. Further related results are due to Wegert [Weg].

8 Levi-Flat Hypersurfaces

At the end of Chapter 23 we saw how Levi-flat hypersurfaces occur in the study of certain polynomial hulls. An existence result in this connection is given by Berndtsson in [B].

9 Polynomial Hulls of Manifolds

We have seen in Theorem 18.7, due to E. Bishop, that certain real manifolds $\Sigma \subseteq \mathbb{C}^n$ contain the boundaries of analytic disks near points $p \in \Sigma$ where the tangent space to Σ at p contains complex linear subspaces of positive dimension. Thus, by the maximum principle, the polynomial hull of Σ contains the corresponding analytic disks. In some cases, these disks form a set that is strictly larger than Σ, and so their presence "explains" the fact that Σ is not polynomially convex.

On the other hand, totally real manifolds by definition do not contain complex linear subspaces of positive dimension in their tangent spaces. These manifolds do not bound "small" analytic disks near a point. In fact, one can show (see [AR], pp. 25–26) that totally real manifolds M are locally polynomially convex in the following sense: For all $x \in M$ and all neighborhoods U of x in M there exists a compact neighborhood K of x in M with $K \subseteq U$ such that K is polynomially convex . Nevertheless, one still wants to "explain" the fact that a totally real manifold M is not polynomially convex by producing analytic disks with boundary in M. For this purpose one can use analytic disks that are smooth up to the unit circle or disks (so-called H^∞ disks) given only by bounded analytic functions whose boundary values (as maps to $\mathbb{C}^n$) exist and lie in M only a.e. on the unit circle. Such disks were given in Example 24.4 for the totally real 3-sphere in $\mathbb{C}^3$ of Ahern and Rudin.

Gromov [Gr] has given a technique to produce analytic disks with boundary in a special class of totally real manifolds of real dimension n in $\mathbb{C}^n$, the Lagrangian manifolds. The classical case of a Lagrangian manifold is one whose tangent space at each point is of the form $\mathcal{U}\mathbb{R}^n$, where $\mathcal{U}$ is a unitary transformation of $\mathbb{C}^n$. (A totally real manifold of real dimension n in $\mathbb{C}^n$ can be described as one whose tangent space at each point is of the form $A\mathbb{R}^n$, where A is a complex linear transformation of $\mathbb{C}^n$.) In [Al5], Gromov's method was adapted to compact orientable totally real manifolds (without boundary) of real dimension n in $\mathbb{C}^n$. In general, nonconstant analytic disks do not exist in this setting, but H^∞ disks do exist.

Recently, Duval and Sibony [DS] have shown, for compact totally real manifolds M in $\mathbb{C}^n$, that rational convexity is equivalent to the existence of certain Kähler forms (which we will not define here) on $\mathbb{C}^n$ that vanish on M. For compact totally real manifolds M of real dimension n in $\mathbb{C}^n$, the existence of these forms is precisely the ("Lagrangian") condition needed in Gromov's theorem. Combining Gromov's theorem and the result of Duval and Sibony, one concludes, in this situation, i.e., for a compact totally real manifold M of real dimension n in $\mathbb{C}^n$, if M is rationally convex, then M bounds a non constant analytic disk. This implies, for example, that a totally real 3-sphere in $\mathbb{C}^3$ is *never* rationally convex (see [DS] Example 3.6).

10 The Polynomial Hull

The most straightforward explanation of why the polynomial hull $\hat{X}$ of a compact set X in $\mathbb{C}^n$ contains a point x is by the maximum principle applied to an analytic disk through x with boundary in X. One step removed from this is a point on an H^∞ disk with boundary in X. Quite different approaches to the hull have been given. Duval and Sibony [DS] discuss the connection between hulls and certain positive currents. Poletsky [Po] has described the polynomial hull of X in terms of pluriharmonic measure. We refer the reader to these papers for the relevant definitions and results.

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Appendix

A1. An account of the theory of subharmonic functions (of one complex variable) can be found in the books by M. Tsuji [Tsu], Chapter 2, and L. Hörmander [Hö2], pp. 16–21. For logarithmic capacity, see [Tsu], Chapter III, or Ransford [Ra], Chapter 5.

A2. We shall require the following result of H. Cartan: *A function subharmonic in a region and equal to $-\infty$ on a Borel set of positive logarithmic capacity is identically $-\infty$ in the region.* For a proof, see [Ra], Theorem 3.5.1.

A3. We are given a plane region U_0, a smooth free boundary arc α of U_0, and a closed subset E of α with $m(E) > 0$, where m is arclength measure. We also are given a function χ bounded and subharmonic on U_0 such that $\limsup_{\lambda\to\lambda_0}\chi(\lambda) = -\infty$ for each $\lambda_0 \in E$. Now fix a point $\lambda_1 \in U_0$. We choose a simply connected region Ω contained in U_0 such that the boundary of Ω is a Jordan curve containing the arc α and $\lambda_1 \in \Omega$. Let Φ be a conformal map of the unit disk D onto Ω with $\Phi(0) = \lambda_1$, and let $\alpha' = \Phi^{-1}(\alpha)$ and $E' = \Phi^{-1}(E)$. Put $f = \chi \circ \Phi$. Then f is subharmonic and bounded on D and $f(z) \to -\infty$ as $z \to \zeta$, for each $\zeta \in E'$.

We identify the unit circle with $[0, 2\pi)$ and E' with a subset of $[0, 2\pi)$. Put $M = \sup f$ over U_0. Then, for each $r < 1$, we have

$$\begin{aligned} f(0) &\le \frac{1}{2\pi}\int_{E'} f(re^{i\theta})\,d\theta + \frac{1}{2\pi}\int_{[0,2\pi)\setminus E'} f(re^{i\theta})\,d\theta \\ &\le \frac{1}{2\pi}\int_{E'} f(re^{i\theta})\,d\theta + M. \end{aligned}$$

As $r \to 1$, $f(re^{i\theta}) \to -\infty$ for each $\theta \in E'$. Since $m(E) > 0$, and Φ is diffeomorphic as a map of α' onto α, $m(E') > 0$. It follows by the bounded convergence theorem that

$$\frac{1}{2\pi}\int_{E'} f(re^{i\theta})\,d\theta \to -\infty$$

as $r \to 1$. Hence $\chi(\lambda_1) = f(0) \leq -\infty$. So $\chi(\lambda_1) = -\infty$. Thus $\chi \equiv -\infty$ on U_0. Therefore, the proposition is proved.

A4. For the disintegration of a measure under a map we refer to the book of Federer [Fe], 2.5.20, where the term decomposition, rather than (Bourbaki's) disintegration, is used.

A5. Pick's Theorem is due to Georg Pick [Pi]. A proof and a discussion of related matters are given in a book by Garnett [Ga], pp. 6–10.

A6. A real-valued function ψ defined on an open set Ω in $\mathbb{C}^n$ is called *plurisubharmonic* on Ω if it is upper semi-continuous and its restriction to each complex line L is subharmonic on $L \cap \Omega$. If ψ belongs to $\mathcal{C}^2$, then ψ is plurisubharmonic on Ω if and only if for each $p \in \Omega$ the inequality

$$\sum_{j,k=1}^{n} \frac{\partial^2 \psi}{\partial z_j \partial \bar{z}_k}(p) \xi_j \bar{\xi}_k \geq 0$$

holds for every vector $(\xi_1, \ldots, \xi_n)$ in $\mathbb{C}^n$. The basic facts about plurisubharmonic functions and their relation to pseudoconvex domains in $\mathbb{C}^n$ are presented in L. Hörmander's book [Hö2], 2.6, and also in the book by R. Gunning [Gu], Vol. 1, Part K.

The "Levi condition" (23.2′) was discovered by E. E. Levi in 1910 [Lev].

A7. Let M and N be smooth manifolds and f a smooth map of M into N. A *critical point* p of the map is a point at which the differential df fails to be surjective as a map between tangent spaces.

Sard's Theorem states that the set of critical values, i.e., $\{f(p) \in N : p$ is a critical point of $f\}$, has measure 0 in N. This result is due to Sard [Sa]; see also J. Milnor [Mi], p. 16. In our application, we take M to be an interval J_j, N to be the unit circle, and f to be the map ψ_j.

A8. We recall that a subvariety V of an open subset Ω of C^n is a closed subset of Ω that is given locally as the set of zeros of a finite set of locally defined analytic functions. For simplicity, we shall restrict our attention to one-dimensional subvarieties. A one-dimensional subvariety V is then a closed subset of Ω such that at each of its points p, except for a discrete "singular" subset of Ω, there are local coordinates $f_1, f_2, \ldots, f_n$ in a neighborhood W of p (i.e., $f = (f_1, f_2, \ldots, f_n)$ map. W biholomorphically to an open subset of C^n) such that $V \cap W = \{z \in W : f_2(z) = 0, f_3(z) = 0, \ldots, f_n(z) = 0\}$. Note that if g is the inverse map of f, then V can be locally parametrized by $\lambda \mapsto g(\lambda, 0, 0, \ldots, 0)$. It follows that the maximum principle holds in the following sense: If F is a holomorphic function on an open subset of C^n whose domain contains a relatively compact open subset W_0 of V, then F attains its maximum modulus over $\bar{W}_0$ on bW_0. A good introduction to this topic can be found in Gunning [Gu], Vol 2., Parts B, C, and D. The book by E. Chirka [Chi1] also can be consulted for a less algebraic approach.

A9. We have defined Runge domain in Chapter 22. The following gives examples of these domains—which we do not assume to be connected here.

Lemma. *Let L be polynomially convex in $\mathbb{C}^n$ and let Ω be the interior of L. Then Ω is a Runge domain.*

PROOF. Let C be a compact subset of Ω. Choose a compact set $C_1 \subseteq \Omega$ such that $C \subseteq \text{int } C_1$. Then $\hat{C} \subseteq \text{int } \hat{C}_1$. (Why?) Since L is polynomially convex, $\hat{C}_1 \subseteq L$ and so $\text{int } \hat{C}_1 \subseteq \text{int } L = \Omega$. Hence $\hat{C} \subseteq \Omega$, and therefore Ω is Runge. □

Every Runge domain is pseudoconvex. If Ω is pseudoconvex, the $\partial\Omega$ is pseudoconvex in the sense of Levi at each point where $\partial\Omega$ is smooth. For these standard implications we refer to Chapter II of [Hö2].

A10. We use a few basic properties of the Hausdorff measure. A very readable, brief presentation of this is given in the paper by Shiffman [Sh]. A comprehensive treatment of this subject is given by Federer [Fe]. We denote the α-dimensional Hausdorff measure of a set Y by $\mathcal{H}^\alpha(Y)$. We list a number of results that can be found in [Sh].

(a) If $\alpha < \beta$ and $\mathcal{H}^\alpha(Y) < \infty$, then $\mathcal{H}^\beta(Y) = 0$.
(b) Let Y be an arbitrary subset of $\mathbb{C}^n$ and let $\alpha > 0$. If $\mathcal{H}^{2k+\alpha}(Y) = 0$, then there exists a complex $(n-k)$-plane P through 0 such that $\mathcal{H}^\alpha(Y \cap P) = 0$.
(c) Let X be a metric space with $a \in X$ and $Y \subseteq X$. Let $S(a, r)$ denote the sphere in X centered at a with radius $r \geq 0$.
Let $\mathcal{H}^1(Y) = 0$. Then $Y \cap S(a, r)$ is empty for almost all r.
(d) Let A be a subset of $\mathbb{R}^n$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^2$ be the projection to the last two coordinates. Suppose that $\mathcal{H}^2(A) < \infty$. Then for almost all points $x \in \mathbb{R}^2$ (with respect to planar measure) $A \cap \pi^{-1}(x)$ is finite.

A11. Recall that for X a compact subset of $\mathbb{C}^n$, $R(X)$ denotes the closure in the uniform norm of the functions on X that are (restrictions of) rational functions with poles not on X. $R(X)$ is a subalgebra of $C(X)$.

Proposition. *$R(X)$ is generated by $n + 1$ functions.*

For the proof, see H. Rossi [Ros2].

A12. We recall that for any two points in the unit disk in $\mathbb{C}$, there is a Möbius transformation of the disk that maps the first point to the second. In Chapter 22, we used the fact that the same is true for the unit ball in $\mathbb{C}^n$. That is, the group of automorphisms (biholomorphic self-mappings) of the unit ball is transitive. For this one can consult Chapter 2 of Rudin's book [Ru3].

A13. We give here the proof of Lemma 19.8, whose statement we recall for the reader's convenience:

Lemma 19.8. $-\bar\partial_\zeta K_1(\zeta, z) = \bar\partial_z K(\zeta, z)$.

PROOF OF LEMMA 19.8. We define functions

$$F(w) = \frac{\bar w_2}{|w|^4} \qquad \text{and} \qquad G(w) = \frac{\bar w_1}{|w|^4},$$

for $w \in \mathbb{C}^2$. Then, suppressing for now the factor $d\zeta_1 \wedge d\zeta_2$,

$$K_1 = F(\zeta - z)\, d\bar z_1 - G(\zeta - z)\, d\bar z_2$$

and

$$K = F(z - \zeta)\, d\bar\zeta_1 - G(z - \zeta)\, d\bar\zeta_2.$$

Since F and G are odd functions of w, it follows that each of the functions $\partial F/\partial\bar w_1$, $\partial F/\partial \bar w_2$, $\partial G/\partial\bar w_1$, $\partial G/\partial\bar w_2$ is an even function of w.

We have

$$\begin{aligned}\bar\partial_\zeta K_1 = &\left(\frac{\partial}{\partial\bar\zeta_1} F(\zeta - z)\, d\bar\zeta_1 + \frac{\partial}{\partial\bar\zeta_2} F(\zeta - z)\, d\bar\zeta_2\right) \wedge d\bar z_1 \\ &- \left(\frac{\partial}{\partial\bar\zeta_1} G(\zeta - z)\, d\bar\zeta_1 + \frac{\partial}{\partial\bar\zeta_2} G(\zeta - z)\, d\bar\zeta_2\right) \wedge d\bar z_2.\end{aligned}$$

Hence

$$\begin{aligned}\bar\partial_\zeta K_1 = &\left(\frac{\partial F}{\partial\bar w_1} (\zeta - z)\, d\bar\zeta_1 + \frac{\partial F}{\partial\bar w_2} (\zeta - z)\, d\bar\zeta_2\right) \wedge d\bar z_1 \\ &- \left(\frac{\partial G}{\partial\bar w_1} (\zeta - z)\, d\bar\zeta_1 + \frac{\partial G}{\partial\bar w_2} (\zeta - z)\, d\bar\zeta_2\right) \wedge d\bar z_2.\end{aligned} \tag{A.1}$$

Further,

$$\begin{aligned}\bar\partial_z K = &\left(\frac{\partial F}{\partial\bar w_1} (z - \zeta)\, d\bar z_1 + \frac{\partial F}{\partial\bar w_2} (z - \zeta)\, d\bar z_2\right) \wedge d\bar\zeta_1 \\ &- \left(\frac{\partial G}{\partial\bar w_1} (z - \zeta)\, d\bar z_1 + \frac{\partial G}{\partial\bar w_2} (z - \zeta)\, d\bar z_2\right) \wedge d\bar\zeta_2.\end{aligned} \tag{A.2}$$

$$\frac{\partial F}{\partial\bar w_1}(w) = \frac{-\bar w_2 \cdot 2|w|^2 w_2}{|w|^8}, \tag{A3a}$$

$$\frac{\partial F}{\partial\bar w_2}(w) = \frac{|w|^4 - \bar w_2 \cdot 2|w|^2 w_2}{|w|^8}, \tag{A3b}$$

$$\frac{\partial G}{\partial\bar w_1}(w) = \frac{|w|^4 - \bar w_1 \cdot 2|w|^2 w_1}{|w|^8}, \tag{A3c}$$

$$\frac{\partial G}{\partial\bar w_2}(w) = \frac{-\bar w_1 \cdot 2|w|^2 w_2}{|w|^8}, \tag{A3d}$$

We next compare the corresponding terms in $(A.1)$ and $(A.2)$. We need to verify that the coefficients of each of the $d\bar\zeta_j \wedge d\bar z_k$ differ only in sign. For the

$d\bar{\zeta}_j \wedge d\bar{z}_j$, $j = 1, 2$, terms, this follows from the fact noted above, that the first partial derivatives of F and G are even functions.

Consider now the $d\bar{\zeta}_2 \wedge d\bar{z}_1$ coefficients. We need to verify that

$$\frac{\partial F}{\partial \bar{w}_2}(\zeta - z) = -\frac{\partial G}{\partial \bar{w}_1}(z - \zeta). \tag{A.4}$$

We note that sum of the right-hand sides of (A.3b) and (A.3c) is identically zero, for all w. Then (A.4) follows by taking $w = \zeta - z$ in this identity. Finally, the $d\bar{\zeta}_1 \wedge d\bar{z}_2$ term is treated in the same way. This completes the proof of Lemma 19.8. □

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Solutions to Some Exercises

Solution to Exercise 3.2. Choose relatively prime polynomials P and Q with $Q \neq 0$ in Ω such that $f = P/Q$. For $t \in C$,

$$\begin{aligned} \frac{f(t) - f(x)}{t - x} &= \frac{Q(x)P(t) - P(x)Q(t)}{Q(t)Q(x)(t - x)} \\ &= \frac{F(x, t)}{Q(t)Q(x)}, \end{aligned}$$

where F is a polynomial in x and t,

$$= \frac{1}{Q(x)} \sum_{j=0}^{N} a_j(t) x^j,$$

where each a_j is holomorphic in Ω. Hence

$$\int_y \frac{f(t) - f(x)}{t - x}\, dt = \frac{1}{Q(x)} \sum_{j=0}^{N} \left\{ \int_\gamma a_j(t)\, dt \right\} x^j = 0,$$

since each a_j is analytic inside γ. Also $\int_\gamma dt/t - x = 2\pi i$. (Why?) Hence the assertion.

Solution to Exercise 9.9. We must prove Theorem 9.7 and so we must show that $\check{S}(\mathcal{L}) \subset X$.

$\check{S}(\mathcal{L})$ is a closed subset of $\mathcal{M}$. Suppose $\exists x_0$ in $\check{S}(\mathcal{L}) \backslash X$. Choose an open neighborhood V of x_0 in $\mathcal{M}$ with $V \cap X = \emptyset$. We may assume that $\bar{V} \subset U_j$ for some j. Since $x_0 \in \check{S}(\mathcal{L})$, $\exists f \in \mathcal{L}$ with

$$\max_{\mathcal{M} \backslash V} |f| < \sup_V |f|,$$

and so

$$\max_{\partial V} |f| < \sup_V |f|.$$

Since $f \in \mathcal{L}$, $\exists f_n \in \mathfrak{A}$ with $f_n \to f$ uniformly on $\bar{V}$. Hence for large n,

$$\max_{\partial V} |f_n| < \sup_V |f_n|.$$

Since $V \subset \mathcal{M} \backslash X$ and $\check{S}(\mathfrak{A}) \subset X$, this contradicts Theorem 9.3. The assertion follows.

Solution to Exercise 17.3 Denote by $x_1, \ldots, x_{2n}$ the real coordinates in $\mathbb{C}^n$. Since a rotation preserves everything of interest to us, we may assume that T is given by

$$x_1 = x_2 = \cdots = x_l = 0, \qquad l = 2n - k.$$

Since $d^2(x) \geq 0$ for all x and $d^2(0) = 0$, we have $\partial(d^2)/\partial x_j = 0$ at $x = 0$ for all j, and so

$$d^2(x) = Q(x) + o(|x|^2),$$

where $Q(x) = \sum_{i,j=1}^{2n} a_{ij} x_i x_j$, $a_{ij} \in \mathbb{R}$. Then

$$Q(x) = \sum_{i,j=1}^{l} a_{ij} x_i x_j + R(x),$$

$R(x)$ being a sum of terms $a_{ij} x_i x_j$ with i or $j > l$. Note that $a_{ij} = a_{ji}$, all i and j.

Assertion. $R = 0$.

We define a bilinear form $[\ ,\ ]$ on $\mathbb{C}^n$ by

$$[x, y] = \sum_{i,j=1}^{2n} a_{ij} x_i y_j.$$

This form is positive semidefinite, since $[x, x] = Q(x) \geq 0$ because $d^2 \geq 0$. Also the form is symmetric, since $a_{ij} = a_{ji}$.

Fix $x^\alpha \in \mathbb{C}^n$ with $x^\alpha = (0, \ldots, 1, \ldots, 0)$, where the 1 is in the αth place and the other entries are 0. Then $[x^\alpha, x^\beta] = a_{\alpha\beta}$. If $\alpha > l$, then $x^\alpha \in T$.

If $x \in T$, then $d^2(s) = o(|x|^2)$, so $Q(x) = 0$. Fix $\alpha > l$. Then $[x^\alpha, x^\alpha] = 0$. It follows that $[x^\alpha, y] = 0$ for all $y \in \mathbb{C}^n$. (Why?) In particular, $a_{\alpha\beta} = [x^\alpha, x^\beta] = 0$ for all β. Hence $R = 0$, as claimed. Thus

$$Q(x) = \sum_{i=1}^{l} a_{ij} x_i x_j. \tag{a}$$

If x is in the orthogonal complement of T and if $|x|$ is small, then the unique nearest point to x on Σ is 0, so $d^2(x) = |x|^2$. Thus if $x = (x_1, x_2, \ldots, x_l, 0, \ldots, 0)$, $d^2(x) = \sum_{i=1}^{l} x_i^2$, so

$$Q(x) = \sum_{i=1}^{l} x_i^2. \tag{b}$$

Equations (a) and (b) yield that

$$Q(x) = \sum_{i=1}^{l} x_i^2$$

for all x. But $\sum_{i=1}^{l} x_i^2 = d^2(x, T)$, So

$$d^2(x) = d^2(x, T) + o(|x|^2).$$

□

Solution to Exercise 18.2 For simplicity, we denote all constants by the same letter C. By hypothesis we have $|h(t)| \leq C|t|^2$ for $t \in \mathbb{R}^N$, $|t| \leq 1$. We regard x as a map from $(0, 2\pi) \to \mathbb{R}^N$. For fixed θ in $(0, 2\pi)$,

$$|h(x(\theta))|^2 \leq C|x(\theta)|^4 \leq C(\|x\|_\infty)^4 < C(\|x\|_1)^4.$$

Hence

$$\int_0^{2\pi} |h(x(\theta))|^2 \, d\theta < C(\|x\|_1)^4. \tag{1}$$

Also $|h_{t_i}(t)| \leq C(t)$ for $|t| \leq 1$. Writing $dx_i/d\theta = \dot{x}_i$, this gives

$$\begin{aligned}\left|\frac{d}{d\theta}(h(x(\theta)))\right| &= \left|\sum_i h_{t_i}(x(\theta))\dot{x}_i(\theta)\right| \\ &\leq \sum_i C|x(\theta)||\dot{x}_i(\theta)| \leq C\|x\|_\infty \sum |\dot{x}_i(\theta)|.\end{aligned}$$

Hence,

$$\left|\frac{d}{d\theta}(h(x(\theta)))\right|^2 \leq C(\|x\|_1)^2 \sum_{i=1}^{N} |\dot{x}_i(\theta)|^2,$$

and so

$$\int_0^{2\pi} \left|\frac{d}{d\theta}(h(x(\theta)))\right|^2 d\theta \leq C\|x\|_1^2 \cdot \|x\|_1^2.$$

(1) and (2) together give $\|h(x)\|_1 \leq C(\|x\|_1)^2$. □

Solution to Exercise 18.3 Fix $t, t' \in \mathbb{R}^N$, $|t| \leq 1$, $|t'| \leq 1$. We claim

$$|h(t) - h(t')| \leq C(|t| + |t'|)|t - t'). \tag{1}$$

For

$$
\begin{aligned}
|h(t') - h(t)| &= \left| \int_0^1 \frac{d}{ds} \{h(t + s(t' - t))\}\, dx \right| \\
&= \left| \int_0^1 \left\{ \sum_{i=1}^{N} h_{t_i}(t + s(t' - t))(t_i' - t_i) \right\} ds \right| \\
&\le \int_0^1 \left\{ \sum_{i=1}^{N} |h_{t_i}(t + s(t' - t))| \right\} |t' - t|\, ds.
\end{aligned}
$$

Also

$$
|h_{t_i}(\zeta)| \le C|\zeta| \qquad \text{for } |\zeta| \le 1.
$$

Hence,

$$
|h(t') - h(t)| \le C(|t| + |t'|)|t' - t|, \text{ i.e., (1).}
$$

Fix θ. By (1)

$$
\begin{aligned}
|h(x(\theta)) - h(y(\theta))| &\le C(|x(\theta)| + |y(\theta)|)(|x(\theta) - y(\theta)|) \\
&\le C(\|x\|_\infty + \|y\|_\infty)(\|x - y\|_\infty) \\
&\le C(\|x\|_1 + \|y\|_1)(\|x - y\|_1).
\end{aligned}
$$

Since this holds for all θ, we have

$$
\|h(x) - h(y)\|_{L^2} \le C(\|x\|_1 + \|y\|_1)(\|x - y\|_1). \tag{2}
$$

Also for fixed θ,

$$
\begin{aligned}
\left| \frac{d}{d\theta} \{h(x) - h(y)\} \right| &= \left| \sum_i h_{t_i}(x)(\dot{x}_i - \dot{y}_i) + \sum_i (h_{t_i}(x) - h_{t_i}(y))\dot{y}_i \right| \\
&\le \sum_i C|x|\, |\dot{x}_i - \dot{y}_i| + \sum_i C|x - y|\, |\dot{y}_i| \\
&\le \sum_i C\|x\|_1 |\dot{x}_i - \dot{y}_i| + \sum_i C\|x - y\|_1 |\dot{y}_i|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\{ \int_0^{2\pi} \left| \frac{d}{d\theta} \{h(x) - h(y)\} \right|^2 d\theta \right\}^{1/2} &\le C\|x\|_1 \sum_i \|\dot{x}_i - \dot{y}_i\|_{L^2} \\
&\quad + C\|x - y\|_1 \sum_i \|\dot{y}_i\|_{L^2} \le C\|x\|_1 \|x - y\|_1 + C\|x - y\|_1 \cdot \|y\|_1.
\end{aligned}
$$

So we have

$$
\left\{ \int_0^{2\pi} \left| \frac{d}{d\theta} \{h(x) - h(y)\} \right|^2 d\theta \right\}^{1/2} \le C(\|x\|_1 + \|y\|_1) \cdot \|x - y\|_1.
$$

Putting (2) and (3) together, we get the assertion. $\square$

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