On Verbal, Pictorial and Constructive Mathematics

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Abstract: One of the classifications of cognitive types which is quite well known in the trichotomy of verbal, pictorial and kinetic. These cognitive styles affect both ways in which mathematicians do dathematics and the ways in which different branches of mathematics are dependent. In fact, there are clear boundaries between verbal mathematics, pictorial Mathematics, and constructive mathematics which we will explore in this paper.

Introduction

Verbal mathematics, pictorial mathematics and constructive mathematics have different tastes and different skills and different approaches toward mathematics. This makes algebra and analysis mostly verbal, and geometry and combinatorics mostly pictorial. Constructive mathematics can be found in all four different branches of algebra, analysis, geometry and combinatorics. Verbal mathematicians learn by simbols; pictorial mathematicians learn by mental images; and constructive mathematicians learn by constructing explicit structures. Mathematical modelling is an important part of constructive mathematics. Even computer modelling of many phenomena is considered as part of constructive mathematics. Differences between cognitive Styles implies differences in mathematics they produce. For example, verbals and kinetics are statistically mostly analytic, and pictorials are statistically mostly holistic. We will discuss the analytic-holistic classification in a forthcoming paper.

On the cognitive type of verbal and on verbal mathematics

Verbal mathematics is full of symbols and is very formal. Formal logic plays an important role in verbal mathematics. Arguments are step by step. They have clear beginning and end. Verbal mathematics can symbolically generalize to case where intuition can not handle. Symbolic similarity between different theories is source of analogy in verbal mathematics. Formalization of mathematics and axiomatic approach to mathematics are both verbal personalities toward mathematics. One can say that all mathematics under the shadow of Euclid's "Element" is verbal. This is a very controversial thing to say. But indeed Euclid's approach to geometry is the verbal formalization of the subject. The verbal philosophy of mathematics is initiated by ideas from Aristotle who himself was a very verbal philosopher of mathematics and philosopher of science are axiomatic and symbolic. Indeed, contribution of Aristotle to logic are base for formal thought and formal proof. One can say that Aristotle is the father of verbal mathematics. Doing mathematics in the verbal style, of course is a general style of doing mathematics but it is bloomed in certain fields of mathematics only. We shall discuss the fields of mathematics which are dominantly verbal in a forthcoming section. This clearly suggests borders between verbal mathematics, and other types of mathematics. These borders are emphasized by the cognitive styles of mathematicians doing mathematics.

On the cognitive type of pictorial and on pictorial math

Intuition and imagination are dominant in pictorial mathematics. Pictorial style of doing mathematics is a completely different type of doing mathematics different from the formal and axiomatic style of verbal mathematics. Arguments of pictorial mathematics are put forward by global visualization. In fact, one is lead to the point to visualize why the result holds and why it should be true. Analogy between different theories in pictorial mathematics is by one-to-one correspondence between concepts and concept relations. It is not necessarily originated by similarity between two formulas like in the verbal mathematics. One can say that intuitive mathematics becomes of the dominance after the book of Euclid. Plato was a pictorial philosopher of cognition. His philosophy of mathematics was also pictorial and existential. Visualizing complicated mathematical facts, was used by Archimedes until Newton and Leibniz. It took 200 years for formalization of mathematical analysis founded by Newton and Leibniz. This style of doing of mathematics bloomed in only some branches of mathematics and defined clear borders between pictorial mathematics and other types of mathematics. These borders are emphasized by the cognitive styles of mathematicians doing mathematics. After the formulation of dichotomy of verbal and pictorial then comes the synthesis of constructive mathematics which is a new type of pictorial mathematics but more precise under the influence of verbal mathematics. We will discuss constructive mathematics in the next section.

On the cognitive type of constructive and on constructive math

Browder in formally known for the idea of constructive mathematics, but the idea goes back to book of “Elements”. Although the style of geometry in the book of Element in axiomatic, but there is certain emphasis on construction. All pictures of geometric theorems of Elements are constructed by ruler and compass. Euclid emphasizes that the shape of a theorem should be drawn and be constructible by certain tools. As if not constructing the shape means that the corresponding mathematics is not real and should not be accepted. Browder officially announced this idea and called it constructive mathematics. These clear borders between verbal mathematics, pictorial mathematics and constructive mathematics. In fact, constructive thought is a certain cognitive type and there is a certain style of doing mathematics called constructive style. The concept of proof in constructive mathematics has to be formal since it is supposed to be constructive, but there is certain intuition involved to see how things should work. It is difficult to decide if constructive mathematics could be both analytic starting from details or holistic by global constructions. For example, square root of 2 is defined holistically and existence of solutions of PDE is naturally proved analytically, and is often constructive.

Algebra is verbal

Algebra is the modern sense of the word, is originated by Descartes, but in its ancient form is invented by the Persian mathematician Al Khwarizmi. In his style of doing algebra, mathematicians wrote equations by words but not by symbols. Although the concept of mathematical symbols was used in arithmetic long before the invention of algebra. We should consider arithmetic also a part of algebra. Also, the ancient field of number theory contains Diophantine equations which are considered part of the verbal field of algebra. The proofs in algebra are analytic and step by step. Algebra concerns itself by details and moves towards getting holistic results at the end. Formal logic plays the main role in algebraic proofs. Algebra over the years has produced many sub branches which are all verbal but not necessarily analytic. Some branches moved towards solving pictorial problem in 20th century but they kept using the same methods of proofs and the same step by step and symbolic nature. Understanding the mathematical concepts in all sub-branches of algebra has never been an intuitive or pictorial task. This is why the problems once reformulated in the language of algebra could be drastically generalized beyond the realm of intuition. The idea of algebraization of mathematics is due to Descartes but is was reawakened by efforts of Hilbert while trying to axiomatize mathematics. It is worth mentioning that the first book of modern algebra was written by Jordan and the concept of algebra as of today is initiated by Noether - early in 20th century.

Analysis is Verbal

Analysis is the same field as algebra except that it has intfinite sums and convergence. So it involves topology which is a verbal field also on top of algebra. Although the proofs are step by step and analytic there are sub branches of analysis which are holistic in approach toward mathematics, like functional analysis. The first instances that infinite sum appeared is, in the works of Archimedes. But Newton and Leibniz where the first mathematicians who used officially infinite sums. It took 200 years to make the foundation of analysis firm under the shadow of Cauchy and Weierstrass and Dedekind. This particular history has caused double standards in the concept of accurate proofs in analysis. There is a concept of intuitive proof after Leibniz which is used to discover new results. But then the analytic step by step proof is used to firmly get the discovered result. The concept of accurate in analysis is by no means pictorial. It is indeed symbolic and verbal. Many branches of analysis are focused on solving problem coming from geometry, but this does not change the nature of arguments in analysis. In the second half of 20^{th} century, there has been attempts to use methods of analysis to solve combinatorial problems, which are also pictorial in nature but this has not changed the nature of arguments in analysis either. But still there is move from verbal algebra and analysis towards pictorial algebra and analysis which the subject of our future exploration.

Moving from verbal to pictorial in algebra

The first step after algebraic treatment of geometric platforms was to start treating algebraic objects as geometric objects. Algebraic varieties are early examples, eventually the concept of scheme and finally non-commutative scheme in the style of Rosenberg appeared, and certainly this is a goal for theorization of the concept of motives. Although the proofs in algebraic geometry are verbal and step by step, verification of proofs and setting the strategic of proofs need considerable intuition and is indeed very pictorial. This has made many pictorial people to get engaged in algebraic geometry and its sub-fields like arithmetic geometry. This has not caused double standards in the proofs since proving theorem in these fields needs both verbal and pictorial skills of cognition. This kind of mathematics needs both analytic and holistic skills of cognition but the proofs are still analytic. Constructing universal objects as ultimate algebraic objects coordinating geometric properties has been the most successful method of algebraic geometry and arithmetic geometry. In rare cases, there is a dictionary between algebraic verbal data and geometry pictorial data which help us to treat the problem with both algebraic and geometric reasonings. Note that the concept of algebraic verbal intuition also exists and is separate from geometric pictorial intuition. Some principles hold when we move from verbal to pictorial analysis. Also, there are both analytic and holistic approaches in pictorial analysis.

Moving from verbal to pictorial in analysis

After we treat problems from geometry and combinatorics using methods of analysis, the first step is to formulate an analytical formulation of objects in combinatorics and geometry. Applications of functional analysis to geometry has progress up to the level where the Connes approach to non-commutative geometry is introduced. There has been earlier approaches before that. For example, people considered the space of continuous complex functions on a topological space and tried to translate topological properties of the space to algebraic properties of space of functions. During the process of picturization of verbal analysis, proofs remain logical and step by step and analytic, but ideas come from geometry which is holistic. There are holistic parts of analysis also for example functional analysis is holistic. To do research in such fields one needs both analytic and pictorial skills of cognition. This has particular implications on the personality of mathematicians performing research in this field. Because, there is confusion if we should move from whole into part or from part into whole. In fact, both happen in functional analysis at the same time. Function is a holistic object and doing analysis on the space of functions is an analytic skill. One usually is lead to treat a function as a local object, for example trying to deform a function locally in a small neighborhood and treat the whole object analytically. An example of doing functional analysis in holistic style is Grothendieck's early research on functional analysis.

Combinatorics is pictorial

Combinatorics is discrete version of geometry and hence pictorial. The problem of counting is also holistic and pictorial. Usually problems in combinatorics can be translated to the language of graph theory which is very much discrete and geometric. There are problems also rooted in combinatorial geometry which are clearly geometric. Many of them has to be solved by discrete optimization methods which are clearly holistic. The proofs are geometric and holistic, and in many instances proof is done by reducing the problem to cases, which is a holistic approach. Many problems in combinatorics have rooted in computer science which are mostly discrete optimization or asymptotic counting. Asymptotic counting could be regarded as moving from pictorial to verbal combinatorics. Although the methods of counting are holistic and intuitive but the problems and ideas coming from verbal mathematics. Asymptotic counting also appears in analytic number theory. Objects and problems are verbal but methods of treatment of problems come from pictorial combinatorics. So there is move from pictorial to verbal appearing in combinatorics also which we will discuss later. In combinatorial geometry sometimes the discrete and continuous aspect of geometric objects is combined so that it's hard to say if we deal this discrete mathematics or continuous mathematics. This means that sometimes the borders between combinatorics and geometry are not clear cuts. Same is for borders between algebra and analysis, where one of them is discrete and the other is continuous. In some places, there are not clear distinctions between them.

Geometry is pictorial

The same way that the field algebra appeared first as an implication of verbal skills of thought, the field geometry appeared as an implication of pictorial skills of thought. The concept of number and arithmetic are the first concepts in verbal mathematics and the concept of shape and its invariants are the first concepts in pictorial mathematics. The concept of a number field or a numerical system appeared much later in 19^{th} century. The same way the concept of space appeared at 17^{th} century after algebraic coordination of geometry by Descartes. Officially Thales was the first to treat geometric objects with verbal and analytic skills of proof. This was much emphasized in Euclid's "Elements". Still this verbal treatment was not able to shadow the intuitive holistic approaches to geometry. Even in the modern concept of geometric space, which is called a manifold, defining the space by charts which is local has not shadowed the intuitive and holistic approaches to these geometric objects. In the field of geometry of manifolds, although many methods are analytic like those coming from analysis, there is tendency to prove global results using these analytic methods. This means that eventually geometry tries to value the holistic understanding of space, rather than analytic and local understanding. Of course, intuition could also be local but not analytic. Like the example of singularities. Note that, there exists a concept of verbal intuition which we will not concern ourselves with that at the moment.

Moving from pictorial to verbal in combinatorics

Verbal treatment of combinatorial problems appears in many different branches. Algebraic combinatorics is one example. There also exist treatments of combinatorial objects using functional analysis or discrete dynamical systems. The methods of proof are verbal and step by step, but doing two kind of mathematics needs both holistic and analytic skills. The same is true for continuous dynamical systems, but we will treat that in the next section. In moving from pictorial to verbal in combinatorics, the problem is proposed from a holistic and intuitive perspective but the solution is introduced using analytic and step by step techniques. Word problem is definitely discrete and is purely verbal. Therefore one can say that there are purely verbal subfields of combinatorics. This is not the case for algebra and analysis. Meaning that there are no purely pictorial subfields of algebra and analysis. Although the word problem can be translated to geometry using Kayley graph of groups, but the methods of solution are all step by step and verbal. Geometric group theory can also be considered as an instance of geometrization of algebra. This is moving from verbal algebra to pictorial graph theory which is discovered in previous section. Word problem is also treated by logic which is purely verbal. In fact, using logic one proves that word problem is undecidable, which is hardly imaginable to be proved geometrically or pictorially. Let we discuss the differences between verbalization of combinatorics and verbalization of geometry.

Moving from pictorial to verbal in Geometry

Verbalization of geometry goes back to long before Descartes. Trigonometry was the first machinery which tried to verbalize spherical and Euclidean geometry. After that using metric formulas was the second verbal method of proof in geometry. Formula like Pythagoras, Heron, and Seva helped to prove geometric theorems using metric formulas. Formulas for area and perimeter of simple shapes had appeared long before that in ancient civilizations. Descartes under the influence of Khayyam had the first modern attempt toward algebraization of geometry. Algebraic geometry dealt with complex solutions of algebraic equations after Abel. Algebraic geometry was very intuitive in the Italian style but because more verbal in the function field formulation by Wiel and Scheme theory by Serre and Grothendieck. Parts of the verbal approach to geometry became the basis of a verbal subfield of algebra called commutative algebra. The methods of proof in the movement from pictorial to verbal became step by step and analytic, and completely changed the nature of the field of geometry. Algebraic topology was the next global attempt to translate geometric data to algebraic structures. Differential calculus was also a method of translating the geometric concepts to analysis which is verbal in its own right. It took 200 years to make a strong mathematical formulation for calculus which was invented by Newton and Leibniz. Algebra, analysis, combinatorics and geometry could also be studied using kinematic or constructive cognitive style which has a different nature and will be studied in the sequel.

On constructive algebra and analysis

Algebra, the way it is understood and communicated to computers is regarded as constructive mathematics. Generators and relations should be explicitly given to or constructed by computer. Every object under consideration by a constructive mathematician should be explicitly computable. There are several software which are capable of such computations. To name a few, Mathematica, Maple, MATLAB and some specialized software like Pari could be noted. Algebra being verbal is the most important aspect of the field which allows it to be dealt using computers. Computers are very verbal tools. Although having constructive does not necessarily mean that we have to be verbal. An important historical example is by construction ruler and compass. Sub-branches of algebra like commutative algebra, algebraic number theory, discrete group theory are most popularly used by computer software. There is tendency to translate all problems in algebra to the language of linear algebra and that could be more easily handled using computers. Actually there are many branches in geometry also that are translated to algebra in order to be dealt with using computer software, like geometric modeling. There are also branches in combinatorial geometry translated to algebra for the same reasoning, like convex geometry. There are parts of algebraic geometry translated to convex geometry to make in computable. This sub-branche is called toric varieties. There is a dictionary between toric varieties and convex bodies. This is a general method in modern mathematics to adapt a sub-branch which is computer friendly to test the conjectures using fast computation.

Many mathematical softwares are calculus friendly. They calculate limits, derivatives, and integrals of standard functions. But as far as I know the most commonly used subfield of analysis which is computer friendly is numerical analysis.

On constructive combinatorics and geometry

Many counting problems are solvable in small scales by computer programming. There is also asymptotic style which is related to computational complexity. Most problems in combinatorics can be translated to the language of graph theory. Graph theory can be easily verbalized and communicated to computers. There are several graph theory software available which could be used for computer modeling of combinatorial problems. These softwares also help to draw graphs and present them pictorially. The subfield of combinatorics which is consisted of finite mathematics is full constructive and can be communicated to computers. Geometric combinatorics on the other hand sometimes cannot be translated to graph theory and hence cannot be dealt with computer software necessarily.

Geometry is essentially intuitive but there has always been approaches in constructing geometric objects since the ruler and compass constructions suggested in Euclid's “Elements”. There have been many popular tools for drawing ellipses and parabolas and hyperbolas but didn't eventually become widely used the way ruler and compass was. Computer modeling of geometric objects is the modern approach to constructive geometry. Although there has been attempts to construct surfaces by clay or chalk on the borders of three dimensional objects. In the computer approach to geometric construction translation to the language of algebra has proven to be very helpful. There has been attempt also to construct software which are capable of proving geometric theorems, which can be regarded as an application of constructive geometry to pure geometry.

On the concept of verbal mathematics and verbal mathematician

Although verbal mathematician have tendency to do verbal mathematics but it is not the case always. This is because verbal mathematics in analytic and verbal mathematicians are statistically analytic mostly. But there are a subset of mathematicians who are verbal but holistic and this causes this subset to more toward pictorial fields. The methods of proof must remain step by step and analytic anyway which is a weakness for this subset. Verbal holistic mathematicians are very much interested in philosophy. They have a formal understanding of history but they understand long term slow phenomena better than local incidences in history. This leads to having a global perspective toward mathematics and being encyclopedia. But generally most verbal mathematicians are analytic and they spend their life doing research in limited particular areas in mathematics. This reminds us of the birds and frogs of Freeman-Dyson. There are certain fields of mathematics which are engaged in answering questions raised by themselves and they are a good target for verbal analytic people. So verbal analytic people sort of limit the methods and tools they use to attack problems in advance which is not the case for verbal holistic mathematicians. This is why most revolutions in verbal mathematics happens because of verbal holistic mathematicians who treat verbal mathematics at the boundaries between different research fields in mathematics. Verbal holistic mathematicians are very similar to pictorial mathematicians.

On the concept of pictorial mathematics and pictorial mathematician

Pictorial mathematicians are statistically holistic. But there is subset of pictorial mathematicians who are analytic and therefore have tendency towards verbal mathematics. The methods of proof for the subset remain intuitive and global. Thus this is sort of disadvantage for pictorial mathematicians to be analytic. Pictorial mathematicians are usually holistic and therefore encyclopedic. But this is not the case for pictorial analytic people. Branches of mathematics which are connected to several other branches are not a good target for pictorial analytic people. The ancient style of doing Euclidean geometry is the most famous example of analytic kind of doing geometry. Modern examples of differential geometry and differential topology, Singularity Theory and any kind of local treatments of geometric objects. Global treatment of geometric objects is dealt by pictorial holistic mathematicians. This is why theorems linking local and global data are of importance. They make a bridge between analytic mathematicians and holistic mathematicians. Pictorial analytic people are engaged also in verbal combinatorics. They try to use the geometric intuition to help their step-by-step analytic thinking. As you see the cognition types of analytic and holistic have serious consequences in the kind of mathematics that mathematicians are engaged in. This will be subject of a further study. Now, we will concentrate on constructive mathematics.

On the concept of constructive Mathematics and constructive mathematician

Constructive mathematician are obsessed with computing. In late 20th century computer programming make this kind of mathematics to flourish. Mathematical modelling and especially computer simulation where the main tools for constructive mathematics. Constructive mathematicians also have a tendency to be holistic. Mathematical modelling for example, models a global phenomenon using mathematical objects. There is a subset of constructive mathematicians which are analytic. These mathematicians are usually engaged in numerical analysis. Local solution of differential equations is a kind of constructive but analytic mathematics. We made this point clear in this paper that different cognitive styles of mathematicians introduces clear borders between the kinds of mathematics they produced. And in any collective study of history of mathematics these borders should be respected and paid attention and studied.