On Theory Oriented, Problem Oriented And Cognition Oriented Mathematics

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Abstract: Problem solving, theorizing and cognitive achievement are three possible motivations for doing mathematics which influences the mathematical personality of mathematicians doing mathematics and also the kind of mathematics they produce. These three ways of doing mathematics have interactions with each other and motivate different concepts of analogy which is the subject of study in this paper.

Introduction

There are branches of mathematics which are problem solving oriented, theorizing oriented, or cognition oriented in the practical philosophy of the mathematical research performed in these branches. Of course, in any field of mathematics research of three kind of mathematicians take part, but one of them only succeeds in management of future development of the field as a whole. This is why we can speak of a branch of mathematics being problem oriented, theory oriented, or cognition. Of course, different branches being holistic or analytic had something to do with being theory oriented or problem oriented. It is more likely for analytic mathematics to be problem oriented and for holistic mathematics to be theory oriented. Becoming cognitive oriented is the third stage which happens to a branch of mathematics after being problem oriented or theory oriented. It is a level of maturity gained by researchers in certain field which overcomes their cognitive style of thinking. We have written other material on cognitive style of doing mathematics elsewhere.

On theory oriented mathematics

In many branches of mathematics, theories developed pretty much explain what is going on in that research field. This is because of the style of doing mathematics by researchers in the field and also the cognitive shape of the material forming that branch of mathematics.

Usually the branches of mathematics which are theory oriented are holistic. Number theory and algebraic geometry are heaven for theoreticians .Also must of holistic field of mathematics are theory oriented. Because this is a good way to treat the material as a whole. The most important strategy in development of theories is using analogies between theories. Undeveloped theories could be further developed using analogies with developed theories. A dictionary between the two theories is most appreciated. Sometimes a dictionary between already developed theories is discovered. This is an indication that there is a truth behind this analogy and the same truth has manifested is different branches or different mathematical situations in mathematics. Theory oriented mathematics is usually done by encyclopedic mathematicians. It reminds me of the birds in “Birds and Frogs” by Freeman- Dyson. Of course there may be problems which are suggested by theories, and there may be theories suggested by problems. In these instances problems are considered part of theories or theory are considered parts of problems, respectively. Sometimes, there is a lack of global theory explaining everything in the field. This is when problem solving becomes dominant.

On problem oriented mathematics

There are branches in mathematics which lack a theoretical framework. They are centered around solving various problems. Most of mathematicians in these fields are analytic and on many occasions verbal. This affects the list of fields in mathematics such that mathematicians have tendency to be problem oriented. Trying to prove theorems in these fields, they try to study important special cases rather than investigating how much the theorem is generalizable and what are obstructions in its generalizable logical arguments have more important roles than intuition and global perspectives. Many mathematicians become interested in mathematics in high school or earlier because of the excitement in problem solving and they continue to keep the excitement alive in their professional carriers too. This is why the majority of mathematicians have tendency to be problem solvers rather than theoreticians. They have to reach a level of maturity to become interested in theorization. They have been very few mathematicians in history that in early age where interested in theorization. Galois is an important and remarkable example against Abel who solved the same problem but did not discover the theoretical framework in the solution. Mathematicians who are interested in theoretical framework are not supported by the mathematical community to be raised as theoreticians. The community of mathematicians only accepts mature and established mathematicians to theorize.

On cognitive mathematics

Theory oriented fields and problem solving oriented field in mathematics eventually reach a level of maturity where neither important problems nor development of theories is the motivating source for developments of those branches of mathematics, but rather than that, cognitive achievements of the research performed by mathematicians is the ruling force for further development of the corresponding branch. Rise of scheme theory as a sub-branch of algebraic geometry leading to appearance of the new field “Arithmetic Geometry” is an example of such a maturization. Advances in field of arithmetic geometry are not motivated by solving problems but by paving the way for new kinds of problems to be solved. Development of theories is also not the motivating force for further development by but finding analogies between theories is an accepted method to go beyond the level of theorization and try to understand the revolutions in the field in the language of cognitive contributions of new ideas to arithmetic geometry. This does not mean that problems which are solved are not regarded to be important or theoretical achievements are not respected, but the direction of the field is determined by cognitive achievements not by important problems or attractive theories. Now it is time to see which branches in mathematics are developed in each category and how the cognitive style of mathematicians influences development of mathematics in each category.

Theory oriented branches

Theory oriented branches are mostly holistic. We named number theory and algebraic geometry as two of the main theory oriented branches of mathematics. Set theory and logic are holistic also and full of important theories, like recursion theory, model theory, proof theory. In combinatorics there are hardly holistic theories. Examples are graph theory and Ramsey theory. In number theory there are several sub-fields in the form of theories like algebraic number theory analytic number theory, multiplicative number theory, probabilistic number theory, computational number theory, field theory, some sub-theories in the field of algebra exists which are holistic like ring theory, and could be considered close to number theory. In algebraic geometry there are also several sub-theories like local theory, (co)homology theory, scheme theory, algebraic groups. Linear algebra contains matrix theory and also representation theory of rings. Category theory and homological algebra and k-theory and groups theory are all parts of holistic algebra. Even many of the subfields of analysis which is not holistic could be theory oriented. Like potential theory, ODE and PDE contains sub-fields like stability theory, qualitative theory, and asymptotic theory, dynamical systems in holistic and contains ergodic theory and bifurcation theory, Fourier analysis and optimization theory and probability theory are also parts of theory oriented mathematics. Geometry and topology and mathematical physics are all theory oriented branches in mathematics. Naming all theories in these pictorial holistic branches needs another attackt.

Problem oriented branches

Some people make fun of the name discrete mathematics and say that it consists of a number of unrelated problems and ideas. Many sub-branches of ODE and PDE are also problem oriented. Applied dynamical systems, like most of other branches of applied mathematics is problem oriented. Problem solving mathematicians are engaged in solving particular problems and they could be working on any field of mathematics. But some of these fields have become mature enough to be guided by development of theories and some get based on solving and generalizing particular problems. Special functions and inequalities are two sub branches of analysis which show this nature in research. Sequences and series, integral equations, numerical analysis and computer science which could be regarded as part of combinatorics have similar nature. mathematicians which are interested in problem oriented branches like to understand a mathematical situation by considering particular cases of the problem which seem to be general enough to show all the difficulties risen in solving the problem. Sometimes it happens that this is the nature of a branch of mathematics and sometimes this gradually change by appearance of guiding theories in that subfield. Further maturity is possible, if a branch of mathematics reaches a level of maturity that cognitive development take the leading role in development of theories.

Cognitive branches

Theory oriented fields are usually centered around a concept, a method of Proof, generalizing a theory to more general setting or on a similar tasks. But in cognitive branches, new methods of cognition and new methods of proof are of importance. Cognitive mathematicians treat mathematics differently. It is a social process if cognitive researchers in a field overcome theory developers and problem solvers or not. This is implemented on the way researchers evaluates research in the field rather than the ways they perform mathematics. In fact, this is about what kind of research is regarded as a breakthrough. Is finding a new analogy between theories a breakthrough in a cognitive branch of mathematics? It depends on the cognitive aspect of the new development. For example, Arakelov theory was a theory of compactifying schemes based on analogy with intersection Theory. But it was considered a breakthrough only because it introduced new ways to understand a geometric space. This is why I believe arithmetic geometry is a cognitive branch of mathematics. I believe that Grothendieck is a cognitive mathematician and his interest in parts of mathematics had cognitive reasons. The theories he develops were not important for him as much as their cognitive contribution were. The concept of “motive” is an example of Grothendieck cognitive approach to mathematics which flourished mathematics for several decades. Most of the cognitive mathematicians I know live under the Shadow of Grothendieck.

Theorizing and problem solving interactions

Problem solving oriented mathematics has serious interaction with theorizing mathematics. Many problems become important because of their role in development of a theory and also many theories are considered achievement because of their role in solving a particular important problem. This is why problem solvers try to translate their problem to a theory to use the cognitive help of theoretical mathematicians and theoretical mathematicians try to introduce problem which help development of their theories in order to use cognitive help of problem solver mathematicians. Usually it is the level of maturity of the mathematician which determines if they are problem solver or theoretician. Although the cognitive structure also plays an important role in being theory or problem oriented in research. This interaction can be regarded as an interaction between holistic and analytic cognitive structures. Birds contribute frogs by showing them methods, analogies and similar problems. And frogs help birds by digging to the depth of the problem. Note that being bird or frog, being holistic and analytic and being theorizer or problem solver are not exactly the same things. But statistically most mathematicians who are birds, are also holistic and theorizer, and most of mathematicians who are frogs are also analytic and problem solvers. It should be emphasized that becoming a theoretical mathematicians needs more maturity than being a problem solver.

Cognitive influence on problem oriented mathematics

Cognition mathematicians influence problem oriented mathematics by evaluating the mathematics produced by a problem solver . For example, if a solution to a problem is illuminating or one should look for another more illuminating solution. What kind of similar problems could be solved by techniques used to solve a particular problem? What kind of techniques used to solve other problems could be used to attach a given problem. How should one generalize the problem or which particular cases of the problem are most important or more illuminating if they are solved? Which analogies between seemingly unrelated problems exist and what kind of problems a particular cognitive structure is appropriate for thinking about? What are the needed cognitive backgrounds for solving a particular problem? What kind of collaborations could help having improvement in solving problems? What is the best way to present a solution to a problem in written format? What would be the appropriate language to do computations in order to solve the problem? There are many philosophical aspects in solving problems and in all those cognitive mathematician could help problem solvers. For more complete list of these philosophical aspects refer to the paper "Who is a good problem solver" by the same author published in proceedings of ICME2011. One should also note that problem solvers also contribute to cognitive mathematics by providing them material to think about.

Cognitive influence on theory oriented mathematics

Cognitive mathematicians also influence Theory oriented mathematics. Analogies between theories could be notified by cognitive mathematics. Analogies could be conceptual or cognitive. Cognitive analogies are hard to be recognized by theoretical mathematicians who are not conscious of cognitive structures. Different kind of theorization have different cognitive backgrounds. For example, there are verbal theories and pictorial theories. Example of verbal theories and pictorial theories are interchangeable. For example, three-manifold Theory is a quite pictorial field, but it can sometimes be translated to verbal mathematics. An important example, is Poincare conjecture being translated to automorphisms of fundamental group which is a problem in discrete group Theory. Cognitive mathematicians are not necessarily holistic. Analytic cognitive mathematician could also exists. It is imaginable that holistic cognitive mathematician influence theorization the most. Analytic cognitive mathematicians influence how theories are related to special cases or special problems. Of course, theory developers are the main group of mathematicians to provide cognitive mathematicians food for thought. Theorizers contribute to cognitive mathematics the most. Analogies between theories which are conceptual are the main contribution of theorizers to cognitive mathematicians. We shall concentrate now, on the concept of analogy from the point of view of theorizers, problem solvers, and cognitive mathematicians.

On analogy between problems

Not all analogies between problems comes from analogies between theories. There are also many analogies between problems which are not cognitive. We study theoretical analogies and cognitive analogies between problems in forthcoming sections. Two problems could be analogous in the trick which they apply to be solved. They can share a common generalization. Assumptions could be close to each other or similar to each other. The kind of statement which is asked to be proved could be very similar. The computational complexity between two problems could be similar and the level of complexity of problems is also a source of similarity. There are mathematicians who as problem solvers try to find problems that are similar to problems they have already solved. They want to look for problems that can be solved by the same tricks or the same mathematics or the same core concepts they are comfortable with. This is the weakest motivation to do mathematics in the ayes of the author of these lines. Some other problem solvers try to extent techniques or replace them by stronger ones to pave the way for other problem solvers. These achievements are also based on analogies between problems. These kinds of mathematicians try to develop techniques to solve several on analogous problems and try to treat them cognitively. This is one level higher in the hierarchy of problem solvers. Eventually problem solvers become abstract enough to try to develop theories to pave the way for other problem solvers and then they will end up developing theories for other purposes.

On analogy between theories

Analogies between theories could be cognitive or conceptual. Finding conceptual analogies is a well-known art which is very popular between theoreticians. Sometimes these are analogies which appear in the form of dictionaries between related or unrelated fields and sometimes they extend to one-to-one correspondence between some objects in different areas of mathematics. Some of these one-to-one correspondence are based on cognitive analogies. For example motives being modular as a part of general Longlands philosophy. We deal with cognitive analogies in the next sections. Dictionaries like Vojta conjectures, Weil conjectures, or Sullivan`s dictionary between Julia sets and Kleinian groups and several others are examples of analogies between theories which are conceptual. Arakelov theory is also another example of theorization based on analogies. There are deep reasons why there are so many dictionaries in number theory and algebraic geometry. This fact make these fields of mathematics heaven for theoreticians. Many cognitive mathematicians are also raised in these two fields. If other mathematicians were raised such a kingdom, they could develop more dictionaries in their own theories. In number theory, the first example of a set of analogous theories appeared. Which lead to the movement of proof of Fermat's last theorem. Analogy was set between modular forms, elliptic curves and Galois representations and many number theorists have devoted their life to extending this trichotomy to more general cases. Many unsolved problems of number theory were settled with the help of extension of this trichotomy.

On cognitive analogy

Translation from one piece of mathematical knowledge to another could be regarded as a conceptual analogy or theorizing analogy also, but the example of Langlands conjectures and the ultimate philosophy that all “motives” are “modular” is much further than just a translation. First of all, it is not a trivial translation, and it’s conjectural and is one of the highest achievements of mathematics to this decade, and now cases are proved by long and technical papers is starting from groundbreaking results of Wiles. The main conjecture goes back to Langlands with ideas set forth by Serre. Shimura and Tanyama had conjectured a special case which was tackled by Wiles and his students and collaborators. Cognitive analogy means that there exists an analogy between cognitive influences of two pieces of mathematics on mathematicians’ cognition. Dichotomy of algebraic and geometric methods and also discrete and continuous methods are the most important examples of cognitive analogy between two realms of cognition. In Langlands program, the Cognitive analogy is between the method of modular forms and the method of algebraic geometry to understand mathematical structures. As a byproduct of the Wiles’ achievements, the oldest unsolved problem in mathematics which has been worked on for 350 years was tackled by Wiles which shows that cognitive analogies is for more effective then theoretical analogies and problem analogies.